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Programa de Pós–Graduação em Matemática

Mestrado em Matemática

Critérios de Regularidade para Soluções Fracas das Equações Magneto-micropolares

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São Cristóvão – SE
Fevereiro de 2016

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sob a orientação do

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A minha mãe, exemplo de força, fé e determinação. Que sempre me apoiou e incentivou aos estudos, e nunca mediu esforços para me ver feliz.

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Resumo

Neste trabalho, discutimos alguns critérios de regularidade para uma solução fraca do sistema de equações tridimensionais de fluido Magneto-micropolar. Além disso, mostramos que é possível estender, para este mesmo sistema, alguns resultados recentes obtidos para as equações de Navier-Stokes. Em ordem a citar um exemplo, provamos que uma solução fraca $(\mathbf{u}, \mathbf{w}, \mathbf{b})(t)$ definida em $[0, T]$ é suave em $\mathbb{R}^3 \times (0, T)$ se esta satisfaz a condição $\partial_3 u_3, \partial_3 \mathbf{w}, \partial_3 \mathbf{b} \in L^\infty(0, T; L^2(\mathbb{R}^3))$.

Palavras-chave: Regularidade de Solução; Equações Magneto-micropolares.

Abstract

This work, we discuss some criteria of regularity for a weak solution of the Magneto-micropolar equations. Furthermore, we show that it is possible to extend some recent results from the Navier-Stokes equations to the Magneto-micropolar equations. In order to give an example, we prove that a weak solution $(\mathbf{u}, \mathbf{w}, \mathbf{b})(t)$, defined in $[0, T]$, is smooth on $\mathbb{R}^3 \times (0, T)$, if it satisfies the condition $\partial_3 u_3, \partial_3 \mathbf{w}, \partial_3 \mathbf{b} \in L^\infty(0, T; L^2(\mathbb{R}^3))$.

Keywords: Regularity of Solutions; Magneto-micropolar Equations.

Sumário

Introdução

Neste trabalho, apresentamos alguns critérios de regularidade para uma solução fraca do seguinte sistema de equações *magneto-micropolar* que descreve um fluido incompressível:

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla(p + \frac{1}{2}|\,\mathbf{b}\,|^2) = (\mu+\chi)\Delta\mathbf{u} + \mathbf{b}\cdot\nabla\mathbf{b} + \chi\nabla\times\mathbf{w}, \\ \mathbf{w}_t + \mathbf{u}\cdot\nabla\mathbf{w} = \gamma\,\Delta\mathbf{w} + \kappa\,\nabla(\nabla\cdot\mathbf{w}) + \chi\nabla\times\mathbf{u} - 2\chi\,\mathbf{w}, \\ \mathbf{b}_t + \mathbf{u}\cdot\nabla\mathbf{b} = \nu\,\Delta\mathbf{b} + \mathbf{b}\cdot\nabla\mathbf{u}, \\ \nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{b} = 0, \\ \mathbf{u}(\cdot,0) = \mathbf{u}_0,\;\; \mathbf{w}(\cdot,0) = \mathbf{w}_0,\;\; \mathbf{b}(\cdot,0) = \mathbf{b}_0, \end{cases} \tag{1}$$

onde $\mathbf{u}(\mathbf{x},t) = (u_1(\mathbf{x},t), u_2(\mathbf{x},t), u_3(\mathbf{x},t)) \in \mathbb{R}^3$ denota o campo velocidade incompressível, $\mathbf{w}(\mathbf{x},t) = (w_1(\mathbf{x},t), w_2(\mathbf{x},t), w_3(\mathbf{x},t)) \in \mathbb{R}^3$ descreve a velocidade micro-rotacional, $\mathbf{b}(\mathbf{x},t) = (b_1(\mathbf{x},t), b_2(\mathbf{x},t), b_3(\mathbf{x},t)) \in \mathbb{R}^3$ o campo magnético e $p(\mathbf{x},t) \in \mathbb{R}$ a pressão. As constantes positivas μ, χ, ν, κ, e γ estão associadas a propriedades específicas do fluido; mais precisamente, μ é a viscosidade cinemática, χ é a viscosidade do vórtice, κ e γ são as viscosidades de rotação e, por último, ν^{-1} é o número magnético de Reynolds. Os dados iniciais para os campos velocidade e magnético, dados por $\mathbf{u}_0$ e $\mathbf{b}_0$ em (1), são assumidos livres de divergente, i.e., $\nabla\cdot\mathbf{u}_0 = \nabla\cdot\mathbf{b}_0 = 0$.

Duas questões relevantes com respeito ao sistema magneto-micropolar (1) podem ser consideradas aqui: a primeira diz respeito à unicidade de soluções fracas e a outra é inerente à regularidade destas mesmas soluções. Com isso em mente, nosso interesse, nesta dissertação, reside em estudar quais hipóteses podemos assumir sobre uma solução fraca de (1) em ordem a obtermos suavidade para esta. Ao leitor mais curioso em compreender a teoria sobre a existência de tais soluções, recomendamos a leitura dos artigos [47, 54] e referências inclusas.

É também importante salientar que o mesmo problema do milênio envolvendo a finitude do tempo máximo de existência para a solução forte do sistema de Navier-Stokes (3) abaixo está também em aberto para o sistema magneto-micropolar (1). Para mais detalhes, envolvendo a solução forte de (1), recomendamos a leitura de [46, 53] e referências inclusas.

Permita-nos informar que os resultados principais discutidos no Capítulos 2 desta dissertação relatam, minuciosamente, as informações obtidas em [1, 61]. Em adição, destacamos que [61] estende o artigo [10], o qual considera regularidade de soluções fracas para o sistema de equações da *magnetohidrodinâmica* (MHD)

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla(p + \frac{1}{2}|\,\mathbf{b}\,|^2) = \mu\Delta\mathbf{u} + \mathbf{b}\cdot\nabla\mathbf{b}, \\ \mathbf{b}_t + \mathbf{u}\cdot\nabla\mathbf{b} = \nu\,\Delta\mathbf{b} + \mathbf{b}\cdot\nabla\mathbf{u}, \\ \nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{b} = 0, \\ \mathbf{u}(\cdot,0) = \mathbf{u}_0(\cdot),\ \ \mathbf{b}(\cdot,0) = \mathbf{b}_0(\cdot). \end{cases} \tag{2}$$

É fácil notar que o sistema acima é derivativo de (1) quando consideramos $\mathbf{w} = 0$ e $\chi = 0$. Outros trabalhos relacionados às equações MHD (2) podem ser encontrados em [4, 5, 10, 12, 14, 21, 24, 27, 28, 29, 32, 37, 40, 41, 55, 63, 64, 65, 71, 83, 84, 85, 86, 87] e referências inclusas.

O artigo [1] também exibe critérios para regularidade de soluções do sistema (1), os quais são inspirados em resultados obtidos para as clássicas equações de *Navier-Stokes*, i.e.,

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mu\Delta\mathbf{u}, \\ \nabla\cdot\mathbf{u} = 0, \\ \mathbf{u}(\cdot,0) = \mathbf{u}_0(\cdot), \end{cases} \tag{3}$$

estas, por sua vez, seguem diretamente de (2) se considerarmos $\mathbf{b}$ como sendo o campo magnético nulo. É importante referir aqui alguns trabalhos que estão relacionados, ou serviram de inspiração, a essa dissertação; sendo assim, veja [2, 6, 7, 8, 9, 11, 13, 15, 17, 18, 19, 22, 23, 25, 26, 30, 31, 34, 35, 36, 38, 39, 42, 43, 45, 48, 49, 50, 51, 52, 56, 57, 58, 59, 60, 68, 69, 70, 72, 73, 74, 75, 76, 78, 79, 80, 81, 82, 88, 89] e referências inclusas.

Resumidamente, gostaríamos de ressaltar que o Capítulo 3 deste trabalho apresenta extensões dos principais resultados obtidos em [68, 73, 74].

A partir de agora, vamos discorrer sobre todo o nosso trabalho com mais detalhes. Assim sendo, estamos interessados em apresentar critérios de regularidade para uma solução fraca do sistema magneto-micropolar (1), quando adotamos condições envolvendo uma componente do campo velocidade ou do seu gradiente. Citamos aqui alguns artigos que estão diretamente relacionados a esta dissertação: [1, 61, 62, 66, 67, 77] e referências inclusas.

No Capítulo 2, estudamos, com detalhes, o artigo [61]. Neste trabalho, Y. Wang mostra que é possível estender suavemente, além de $t = T$, uma solução fraca $(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t)$ de (1), definida no intervalo de tempo $[0, T]$, quando lhe é imposta a seguinte condição sobre uma componente do gradiente do campo velocidade $\mathbf{u}(\cdot, t)$:

$$\int_0^T \|\partial_3 \mathbf{u}(\cdot, t)\|_p^q \, dt < \infty, \quad \frac{3}{p} + \frac{2}{q} \leq 1, \, p \geq 3.$$

Vejamos, com mais detalhes, o teorema que garante a afirmação dada acima.

Teorema 0.1 (ver [61]). *Seja* $(\mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0) \in H^1(\mathbb{R}^3)$ *tal que* $\nabla \cdot \mathbf{u}_0 = \nabla \cdot \mathbf{b}_0 = 0$. *Assuma que* $(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t)$ *é uma solução fraca para o sistema magneto-micropolar* (1) *no intervalo* $[0, T]$. *Se*

$$\int_0^T \|\partial_3 \mathbf{u}(\cdot, t)\|_p^q \, dt < \infty, \quad \frac{3}{p} + \frac{2}{q} \leq 1, \, p \geq 3,$$

então a solução $(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t)$ *pode ser estendida suavemente além de* $t = T$.

Ainda no Capítulo 2, exibimos minuciosamente dois dos resultados obtidos por Y. Baoquan [1]. Tais resultados, também tratam da regularidade de uma solução fraca $(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t)$ das equações magneto-micropolares (1). No primeiro deles, Y. Baoquan provou que tal solução, definida em $[0, T]$, pode ser suavemente estendida além de $t = T$, quando adotamos a seguinte condição envolvendo somente o campo velocidade $\mathbf{u}(\cdot, t)$:

$$\mathbf{u} \in L^q(0, T; L^p(\mathbb{R}^3)), \quad \frac{3}{p} + \frac{2}{q} \leq 1, \, 3 < p \leq \infty.$$

Mais precisamente, vejamos como a afirmação acima pode ser enunciada.

Teorema 0.2 (ver [1]). *Seja* $(\mathbf{u_0}, \mathbf{w_0}, \mathbf{b_0}) \in H^1(\mathbb{R}^3)$ *tal que* $\nabla \cdot \mathbf{u}_0 = \nabla \cdot \mathbf{b}_0 = 0$. *Assuma que* $(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t) \in C[0, T; H^1(\mathbb{R}^3)) \cap C(0, T; H^2(\mathbb{R}^3))$ *é uma solução suave para o sistema* (1). *Se* $(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t)$ *satisfaz*

$$\mathbf{u} \in L^q(0, T; L^p(\mathbb{R}^3)), \quad \frac{3}{p} + \frac{2}{q} \leq 1, \, 3 < p \leq \infty,$$

então a solução $(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t)$ *pode ser suavemente estendida além de* $t = T$.

Ainda por [1], estabelecendo determinada exigência sobre o gradiente do campo velocidade, $\nabla \mathbf{u}(\cdot, t)$, vimos detalhadamente como estender suavemente além de $t = T$, uma solução fraca

$(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t)$, definida em $[0, T]$, de (1). Segue a exigência necessária para a obtenção do resultado.

$$\int_0^T \|\nabla \mathbf{u}(\cdot, t)\|_p^q \, dt < \infty, \quad \frac{3}{p} + \frac{2}{q} \leq 2, \; \frac{3}{2} < p \leq \infty.$$

Vejamos, especificamente, como se dá o enunciado de tal afirmação.

Teorema 0.3 (ver [1])**.** *Seja* $(\mathbf{u_0}, \mathbf{w_0}, \mathbf{b_0}) \in H^1(\mathbb{R}^3)$ *tal que* $\nabla \cdot \mathbf{u}_0 = \nabla \cdot \mathbf{b}_0 = 0$. *Assuma que* $(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t) \in C[0, T; H^1(\mathbb{R}^3)) \cap C(0, T; H^2(\mathbb{R}^3))$ *é uma solução suave para o sistema* (1). *Se* $(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t)$ *satisfaz*

$$\int_0^T \|\nabla \mathbf{u}(\cdot, t)\|_p^q \, dt < \infty, \quad \frac{3}{p} + \frac{2}{q} \leq 2, \; \frac{3}{2} < p \leq \infty.$$

então a solução $(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t)$ *pode ser suavemente estendida além de* $t = T$.

Os Teoremas 0.1, 0.2 e 0.3 serão estudados com detalhes nas Seções 2.1, 2.2 e 2.3 desta dissertação.

No Capítulo 3, apresentamos extensões, no contexto das equações magneto-micropolares (1), dos resultados obtidos em [68, 73, 74]. Ou seja, conseguimos provar que uma solução fraca $(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t)$ de (1) é suave se adotarmos a seguinte hipótese:

$$(\nabla u_3, \nabla_h \mathbf{w}, \nabla_h \mathbf{b}) \in L^{\frac{32}{7}}(0, T; L^2(\mathbb{R}^3)).$$

Mais precisamente, demonstramos o seguinte teorema:

Teorema 0.4. *Sejam* $(\mathbf{u_0}, \mathbf{w_0}, \mathbf{b_0}) \in L^2(\mathbb{R}^3)$ *com* $\nabla \cdot \mathbf{u_0} = \nabla \cdot \mathbf{b_0} = 0$. *Assuma que* $T > 0$. *Considere que* $(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t)$ *é uma solução fraca das equações magneto-micropolares* (1) *em* $[0, T]$ *com condição inicial* $(\mathbf{u_0}, \mathbf{w_0}, \mathbf{b_0})$. *Se,*

$$(\nabla u_3, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})(\cdot, t) \in L^{\frac{32}{7}}(0, T; L^2(\mathbb{R}^3)),$$

então $(\mathbf{u}, \mathbf{w}, \mathbf{b})$ *é suave em* $\mathbb{R}^3 \times (0, T)$.

O critério dado no teorema acima foi provado em [73], se considerarmos o caso particular das equações de Navier-Stokes. Mais claramente, Z. Zhang e X. Yang [73] estabeleceram que uma solução fraca $\mathbf{u}(\cdot, t)$, definida em $[0, T]$, é suave em $\mathbb{R}^3 \times (0, T)$ se é assumida a seguinte hipótese sobre o gradiente da terceira componente do campo velocidade $\mathbf{u}(\cdot, t)$:

$$\nabla u_3 \in L^{\frac{32}{7}}(0, T; L^2(\mathbb{R}^3)).$$

Para mais detalhes ver [73].

Ao estudarmos Z. Zhang e X. Yang [74], é possível nos depararmos com outro critério de regularidade para uma solução fraca $\mathbf{u}(\cdot,t)$, definida em $[0,T]$, das equações de Navier-Stokes (3). Este é descrito da seguinte forma:

$$\partial_3 u_3 \in L^\infty(0,T;L^2(\mathbb{R}^3)).$$

(Para mais detalhes ver [74]). De forma análoga ao desenvolvimento dado em [74], garantimos a suavidade de uma solução fraca $(\mathbf{u},\mathbf{w},\mathbf{b})(\cdot,t)$, definida em $[0,T]$, do sistema magneto-micropolar (1), exigindo a condição abaixo:

$$(\partial_3 u_3, \partial_3 \mathbf{w}, \partial_3 \mathbf{b}) \in L^\infty(0,T;L^2(\mathbb{R}^3)).$$

Escrevendo a afirmação acima em forma de teorema, temos o seguinte:

Teorema 0.5. *Sejam* $(\mathbf{u_0},\mathbf{w_0},\mathbf{b_0}) \in L^2(\mathbb{R}^3)$ *com* $\nabla\cdot\mathbf{u_0} = \nabla\cdot\mathbf{b_0} = 0$. *Assuma que* $T>0$. *Considere que* $(\mathbf{u},\mathbf{w},\mathbf{b})(\cdot,t)$ *é uma solução fraca das equações magneto-micropolares* (1) *em* $[0,T]$ *com condição inicial* $(\mathbf{u_0},\mathbf{w_0},\mathbf{b_0})$. *Se,*

$$(\partial_3 u_3, \partial_3 \mathbf{w}, \partial_3 \mathbf{b})(\cdot,t) \in L^\infty(0,T;L^2(\mathbb{R}^3)),$$

então $(\mathbf{u},\mathbf{w},\mathbf{b})$ *é suave em* $\mathbb{R}^3\times(0,T)$.

Por fim, Z. Zhang [68] mostrou que uma solução fraca $\mathbf{u}(\cdot,t)$, definida em $[0,T]$, do sistema de Navier-Stokes (3) é suave em $\mathbb{R}^3\times(0,T)$, quando

$$\nabla u_3 \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{3}{q}+\frac{2}{p}=\frac{3}{2}+\frac{3}{4q},\ \frac{3}{2}<q<3.$$

(Para mais detalhes ver [68]). A extensão de [68], obtida nesta dissertação, para o contexto das equações magneto-micropolares (1), é dada como segue.

Teorema 0.6. *Seja* $(\mathbf{u_0},\mathbf{w_0},\mathbf{b_0}) \in L^2(\mathbb{R}^3)$ *com* $\nabla\cdot\mathbf{u_0} = \nabla\cdot\mathbf{b_0} = 0$ *e* $T>0$. *Assuma que* $(\mathbf{u},\mathbf{w},\mathbf{b})(\cdot,t)$ *é uma solução fraca das equações magneto-micropolares* (1) *em* $[0,T]$ *com condição inicial* $(\mathbf{u_0},\mathbf{w_0},\mathbf{b_0})$. *Se,*

$$(\nabla u_3, \nabla\mathbf{w}, \nabla\mathbf{b})(\cdot,t) \in L^p(0,T;L^q(\mathbb{R}^3)); \quad \frac{3}{q}+\frac{2}{p}=\frac{3}{2}+\frac{3}{4q},\ \frac{3}{2}<q<3,$$

então a solução $(\mathbf{u},\mathbf{w},\mathbf{b})$ *é suave em* $\mathbb{R}^3\times(0,T)$.

Todas as extensões citadas acima, estão devidamente detalhadas, nas Seções 3.1, 3.2 e 3.3 dessa dissertação.

É importante destacar aqui que todas as provas dos resultados principais desta dissertação consideram derivadas clássicas da solução fraca $(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t)$ para (1). A existência de tais derivadas seguem do fato que quando comparamos o tempo maximal T^* para a existência da solução forte do sistema (1) com T (ver Teoremas 0.1, 0.2, 0.3, 0.4, 0.5 e 0.6) só precisamos analisar o caso $T^* < T$, já que a outra condição nos leva, de forma trivial, à desejada regularidade da solução fraca (esta, neste caso, será a solução forte).

O esboço do restante do trabalho é dado como segue: no Capítulo 1 apresentamos as notações, as definições e os resultados básicos necessários para um bom entendimento do conteúdo; no Capítulo 2, as provas dos resultados dados em [1, 61] são estabelecidas; e, por último, no Capítulo 3, acrescentamos as extensões obtidas a partir dos artigos [68, 73, 74].

Capítulo 1

Notações, Definições e Resultados Preliminares

Neste capítulo, listamos algumas notações e definições que serão aplicadas em todo o trabalho. Aproveitamos também para acrescentar, sem provas, alguns resultados bem conhecidos que serão utilizados aqui como, por exemplo, Desigualdades de Hölder e Gagliardo-Nirenberg.

1.1 Notações e Definições

Primeiramente, vamos apresentar algumas notações e definições que serão utilizadas ao longo dessa dissertação.

- Letras em negrito denotam campos de vetores, como por exemplo,

$$\mathbf{a} = \mathbf{a}(x,t) = (a_1(x,t), a_2(x,t), a_3(x,t)), \quad x \in \mathbb{R}^3,\ t \geq 0.$$

- A norma Euclidiana de qualquer vetor $\mathbf{a} = (a_1, ..., a_n) \in \mathbb{R}^n$ é denotada por $|\mathbf{a}| = \sqrt{\sum_{i=1}^n a_i^2}$.
- A notação $L^\alpha(\mathbb{R}^3;\mathbb{R}^3)$ é utilizado para o espaço de Lebesgue equipado com a norma $\|\cdot\|_\alpha$, onde $1 \leq \alpha \leq \infty$; mais especificamente,

$$\|\mathbf{a}\|_\alpha := \left(\int_{\mathbb{R}^3} |\mathbf{a}(x)|^\alpha \, dx\right)^{\frac{1}{\alpha}}, \quad 1 \leq \alpha < \infty,$$

e

$$\|\mathbf{a}\|_\infty := \sup_{x\in\mathbb{R}^3} ess\,\{|\mathbf{a}(x)|\},$$

onde $\mathbf{a}: \mathbb{R}^3 \to \mathbb{R}^n$ $(n \in \mathbb{N})$ é uma função mensurável. Definimos o produto interno em L^2 de duas funções vetoriais por

$$(\mathbf{a}, \mathbf{b})_2 := \int_{\mathbb{R}^3} \mathbf{a}(x) \cdot \mathbf{b}(x)\, dx,$$

onde $\mathbf{c} \cdot \mathbf{d} := \sum_{i=1}^n c_i d_i$ para $\mathbf{c} = (c_1, ..., c_n), \mathbf{d} = (d_1, ..., d_n) \in \mathbb{R}^n$; e $\mathbf{a}, \mathbf{b} : \mathbb{R}^3 \to \mathbb{R}^n$ $(n \in \mathbb{N})$ são funções mensuráveis.

- Considere $\nabla \mathbf{a} = (\nabla a_1, ..., \nabla a_n)$ a notação para o gradiente de $\mathbf{a} = (a_1, ..., a_n) \in \mathbb{R}^n$, onde $\nabla a_j = (\partial_1 a_j, \partial_2 a_j, \partial_3 a_j)$, com $\partial_i = \partial/\partial x_i$ para todo $i = 1, 2, 3$ e $j = 1, ..., n$.
- O gradiente horizontal é denotado por $\nabla_h \mathbf{a} = (\nabla_h a_1, ..., \nabla_h a_n)$, onde $\mathbf{a} = (a_1, ..., a_n) \in \mathbb{R}^n$ e $\nabla_h a_j = (\partial_1 a_j, \partial_2 a_j)$, com $j = 1, ..., n$.
- Aqui $\mathbf{a} \cdot \nabla := \sum_{i=1}^3 a_i \partial_i$, onde $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$.
- $\nabla \times \mathbf{a}$ indica o rotacional de $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$.
- Denote $\nabla \cdot \mathbf{a} = \sum_{i=1}^3 \partial_i a_i$, onde $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$.
- Δ representa o operador Laplaciano.
- O Laplaciano horizontal é denotado por $\Delta_h \mathbf{a} = (\Delta_h a_1, ..., \Delta_h a_n)$, onde $\mathbf{a} = (a_1, ..., a_n) \in \mathbb{R}^n$ e $\Delta_h a_j = \sum_{i=1}^2 \partial_i^2 a_j$, com $j = 1, ..., n$.
- O espaço de Sobolev $H^1(\mathbb{R}^3)$ é munido da norma

  $$\|\mathbf{a}\|_{1,2} := \sqrt{\|\mathbf{a}\|_2^2 + \|\nabla \mathbf{a}\|_2^2},$$

  onde $\mathbf{a}: \mathbb{R}^3 \to \mathbb{R}^n$ $(n \in \mathbb{N})$ é uma função mensurável.
- Seja $(X, \|\cdot\|)$ um espaço de Banach e assuma que $1 \le \beta \le \infty$, $c, d \in \mathbb{R}$. Denotamos por $L^\beta(c, d; X)$ o espaço de todas as funções mensuráveis $f : [c, d] \to X$ com $\|f(\cdot)\| \in L^\beta([c, d])$ dotado com a norma $\|f\|_{L^\beta(c,d;X)} := \left(\int_c^d \|f(t)\|^\beta\, dt\right)^{\frac{1}{\beta}}$, onde $\beta < \infty$; e também $\|f\|_{L^\infty(c,d;X)} = \sup_{t\in[c,d]} ess\,\{\|f(t)\|\}$.
- $C_c^1(\mathbb{R}^n)$ denota o espaço das funções de classe $C^1(\mathbb{R}^n)$ com suporte compacto em $\mathbb{R}^n$.

- $C_c^\infty(\mathbb{R}^n)$ denota o espaço das funções suaves com suporte compacto em $\mathbb{R}^n$.

- Definimos uma solução fraca de (1) do seguinte modo: Seja $T > 0$ e $(\mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0) \in L^2(\mathbb{R}^3)$, com $\nabla \cdot \mathbf{u}_0 = \nabla \cdot \mathbf{b}_0 = 0$. A função mensurável $(\mathbf{u}, \mathbf{w}, \mathbf{b})(x,t)$ chama-se uma solução fraca de (1) em $[0,T)$ se as seguintes condições são garantidas

  1. $(\mathbf{u}, \mathbf{w}, \mathbf{b})(x,t) \in L^\infty(0,T; L^2(\mathbb{R}^3)) \cap L^2(0,T; H^1(\mathbb{R}^3))$;
  2. o sistema (1) é satisfeito no sentindo de distribuições;
  3. a desigualdade de energia é válida, isto é,

  $$\begin{aligned}&\|(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot,t)\|_2^2 + 2(\mu+\chi)\int_0^t \|\nabla \mathbf{u}(\cdot,\tau)\|_2^2\, d\tau + 2\gamma \int_0^t \|\nabla \mathbf{w}(\cdot,\tau)\|_2^2\, d\tau \\ &+ 2\nu \int_0^t \|\nabla \mathbf{b}(\cdot,\tau)\|_2^2\, d\tau + 2\kappa \int_0^t \|\nabla \cdot \mathbf{w}(\cdot,\tau)\|_2^2\, d\tau + 2\chi \int_0^t \|\mathbf{w}(\cdot,\tau)\|_2^2\, d\tau \\ &\leq \|(\mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0)\|_2^2, \end{aligned} \tag{1.1}$$

  para todo $0 \leq t < T$.

- As constantes, neste trabalho, podem mudar linha por linha, mas serão denotadas da mesma maneira. Também, em alguns momentos, não nos preocuparemos com a notação de dependência em x e t, por exemplo, $\|\mathbf{u}\|_2$ ou $\|\mathbf{u}(t)\|_2$ significará $\|\mathbf{u}(\cdot,t)\|_2$.

1.2 Resultados Preliminares

Nesta seção, apresentaremos alguns resultados que serão aplicados com bastante frequência neste trabalho.

- (Desigualdade de Hölder)(ver [20]): Sejam $1 \leq p, q \leq \infty$ expoentes conjugados, isto é, $\frac{1}{p} + \frac{1}{q} = 1$. Sejam $f : \mathbb{R}^3 \to \mathbb{R}$ e $g : \mathbb{R}^3 \to \mathbb{R}$ funções tais que $f \in L^p(\mathbb{R}^3)$ e $g \in L^q(\mathbb{R}^3)$. Então,

  $$\int_{\mathbb{R}^3} |f(x)g(x)|\, dx \leq \|f\|_p \|g\|_q.$$

- (Desigualdade de Young)(ver [20]): Se $1 < p, q < \infty$ tais que $\frac{1}{p} + \frac{1}{q} = 1$, então para todo par de números reais a e b não negativos, vale a desigualdade

  $$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

- (Desigualdade de Gronwall)(ver [16]): Seja $f(t)$ uma função não negativa e diferenciável em $[0,T]$, que satisfaz:

$$f'(t) \leq g(t)f(t) + h(t), \quad \forall\, t \in [0,T],$$

onde $g(t)$ e $h(t)$ são funções contínuas não negativas em $[0,T]$. Então,

$$f(t) \leq e^{\int_0^t g(\tau)\,d\tau}[f(0) + \int_0^t h(\tau)\,d\tau], \quad \forall\, t \in [0,T].$$

- (Desigualdade de Gagliardo-Nirenberg)(ver [44]): Sejam $1 \leq p < n$, $1 \leq s < \infty$ e $0 \leq \theta \leq 1$. Existe uma constante positiva C tal que

$$\|\mathbf{u}\|_q \leq C\|\mathbf{u}\|_s^{1-\theta}\|\nabla\mathbf{u}\|_p^{\theta}, \tag{1.2}$$

onde $\frac{1}{q} = \theta\left(\frac{1}{p} - \frac{1}{n}\right) + \frac{1-\theta}{s}$.

- (Desigualdade de Interpolação)(ver [20]): Sejam $0 < p < q < r \leq \infty$ e $\mathbf{u} \in L^p \cap L^r$. Então,

$$\|\mathbf{u}\|_q \leq \|\mathbf{u}\|_p^{\theta}\|\mathbf{u}\|_r^{1-\theta},$$

com $\theta = \frac{q^{-1}-r^{-1}}{p^{-1}-r^{-1}}$.

Capítulo 2

Critérios de Regularidade Envolvendo Somente o Campo Velocidade

Neste capítulo, estudaremos, minuciosamente, o resultado principal apresentado em [61]. Mais precisamente, mostraremos como Y. Wang [61] provou que uma solução fraca $(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t)$, definida em $[0, T]$, do sistema magneto-micropolar (1) pode ser estendida suavemente além de T se

$$\int_0^T \|\partial_3 \mathbf{u}(\cdot, t)\|_p^q \, dt < \infty, \quad \frac{3}{p} + \frac{2}{q} \leq 1, \, p \geq 3.$$

A extensão relatada acima, também discutida aqui, foi provada por Y. Baoquan [1] quando foi assumida uma duas hipósteses abaixo:

$$\mathbf{u}(\cdot, t) \in L^q(0, T; L^p(\mathbb{R}^3)), \quad \frac{3}{p} + \frac{2}{q} \leq 1, \, 3 < p \leq \infty,$$

ou ainda

$$\int_0^T \|\nabla \mathbf{u}(\cdot, t)\|_p^q \, dt < \infty, \quad \frac{3}{p} + \frac{2}{q} \leq 2, \, \frac{3}{2} < p \leq \infty.$$

2.1 Critério de Regularidade Envolvendo Somente $\partial_3 \mathbf{u}(\cdot, t)$

Nesta seção, nossa meta é provar que uma solução fraca $(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t)$ do sistema magneto-micropolar (1), definida em $[0, T]$, pode ser estendida suavemente além de T, quando a seguinte hipótese, envolvendo somente uma componente do gradiente do campo velocidade $\mathbf{u}(\cdot,t)$, é imposta:

$$\int_0^T \|\partial_3 \mathbf{u}\|_p^q \, dt < \infty, \quad \frac{3}{p} + \frac{2}{q} \leq 1,\, p \geq 3.$$

Antes de enunciar o resultado que garante o cumprimento de nosso objetivo, permita-nos estabelecer dois lemas que desempenham um papel importante na prova do Teorema 2.1 abaixo. O primeiro deles foi provado em [10].

Lema 2.1 (ver [10]). *Assuma que $\theta, \lambda, \vartheta \in \mathbb{R}$ satisfazem*

$$1 \leq \theta, \lambda < \infty,\, \frac{1}{\theta} + \frac{2}{\lambda} > 1,\, 1 + \frac{3}{\vartheta} = \frac{1}{\theta} + \frac{2}{\lambda}.$$

Assuma que $f \in H^1(\mathbb{R}^3)$, $\partial_1 f, \partial_2 f \in L^\lambda(\mathbb{R}^3)$ e $\partial_3 f \in L^\theta(\mathbb{R}^3)$. Então, existe uma constante positiva C tal que

$$\|f\|_\vartheta \leq C \|\partial_1 f\|_\lambda^{\frac{1}{3}} \|\partial_2 f\|_\lambda^{\frac{1}{3}} \|\partial_3 f\|_\theta^{\frac{1}{3}}.$$

Particularmente, se $\lambda = 2$ e $f \in H^1(\mathbb{R}^3)$, $\partial_3 f \in L^\theta(\mathbb{R}^3)$ (com $1 \leq \theta < \infty$), então existe uma constante positiva C tal que

$$\|f\|_{3\theta} \leq C \|\partial_1 f\|_2^{\frac{1}{3}} \|\partial_2 f\|_2^{\frac{1}{3}} \|\partial_3 f\|_\theta^{\frac{1}{3}}. \tag{2.1}$$

Demonstração. Considere, sem perda de generalidade, que $f \in C_c^\infty(\mathbb{R}^3)$. Assim sendo, note que

$$\int_{-\infty}^{x_1} \partial_1 \{[f(s,x_2,x_3)]^{1+\left(1-\frac{1}{\lambda}\right)\vartheta}\}\, ds = [1 + \left(1 - \frac{1}{\lambda}\right)\vartheta] \int_{-\infty}^{x_1} [f(s,x_2,x_3)]^{\left(1-\frac{1}{\lambda}\right)\vartheta} \partial_1 f(s,x_2,x_3)\, ds.$$

Assim, pelo Teorema Fundamental do Cálculo, temos que

$$\begin{aligned} |f(x)|^{1+\left(1-\frac{1}{\lambda}\right)\vartheta} &\leq C \int_{-\infty}^{x_1} |f(s,x_2,x_3)|^{\left(1-\frac{1}{\lambda}\right)\vartheta} \, |\partial_1 f(s,x_2,x_3)| \, ds \\ &\leq C \int_{\mathbb{R}} |f(x_1,x_2,x_3)|^{\left(1-\frac{1}{\lambda}\right)\vartheta} \, |\partial_1 f(x_1,x_2,x_3)| \, dx_1. \end{aligned}$$

Analogamente, conclui-se que

$$|f(x)|^{1+\left(1-\frac{1}{\lambda}\right)\vartheta} \leq C \int_{\mathbb{R}} |f(x_1,x_2,x_3)|^{\left(1-\frac{1}{\lambda}\right)\vartheta} \, |\partial_2 f(x_1,x_2,x_3)| \, dx_2$$

e também

$$|f(x)|^{1+\left(1-\frac{1}{\theta}\right)\vartheta} \leq C \int_{\mathbb{R}} |f(x_1,x_2,x_3)|^{\left(1-\frac{1}{\theta}\right)\vartheta} \, |\partial_3 f(x_1,x_2,x_3)| \, dx_3.$$

Desta forma, chegamos a

$$
\begin{aligned}
|f(x)|^{\vartheta} \leq C &\left(\int_{\mathbb{R}} |f(x_1, x_2, x_3)|^{\left(1-\frac{1}{\lambda}\right)\vartheta} |\partial_1 f(x_1, x_2, x_3)| \, dx_1 \right)^{\frac{1}{2}} \\
&\times \left(\int_{\mathbb{R}} |f(x_1, x_2, x_3)|^{\left(1-\frac{1}{\lambda}\right)\vartheta} |\partial_2 f(x_1, x_2, x_3)| \, dx_2 \right)^{\frac{1}{2}} \\
&\times \left(\int_{\mathbb{R}} |f(x_1, x_2, x_3)|^{\left(1-\frac{1}{\theta}\right)\vartheta} |\partial_3 f(x, y, z)| \, dx_3 \right)^{\frac{1}{2}}.
\end{aligned}
$$

Integrando com respeito a x_1, chegamos a

$$
\begin{aligned}
\int_{\mathbb{R}} |f(x_1, x_2, x_3)|^{\vartheta} \, dx_1 \leq C &\left(\int_{\mathbb{R}} |f(x_1, x_2, x_3)|^{\left(1-\frac{1}{\lambda}\right)\vartheta} |\partial_1 f(x_1, x_2, x_3)| \, dx_1 \right)^{\frac{1}{2}} \\
&\times \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x_1, x_2, x_3)|^{\left(1-\frac{1}{\lambda}\right)\vartheta} |\partial_2 f(x_1, x_2, x_3)| \, dx_2 \right)^{\frac{1}{2}} \\
&\times \left(\int_{\mathbb{R}} |f(x_1, x_2, x_3)|^{\left(1-\frac{1}{\theta}\right)\vartheta} |\partial_3 f(x_1, x_2, x_3)| \, dx_3 \right)^{\frac{1}{2}} dx_1.
\end{aligned}
$$

Portanto, pela Desigualdade de Hölder, podemos escrever o seguinte:

$$
\begin{aligned}
\int_{\mathbb{R}} |f(x_1, x_2, x_3)|^{\vartheta} \, dx_1 \leq C &\left(\int_{\mathbb{R}} |f(x_1, x_2, x_3)|^{\left(1-\frac{1}{\lambda}\right)\vartheta} |\partial_1 f(x_1, x_2, x_3)| \, dx_1 \right)^{\frac{1}{2}} \\
&\times \left(\int_{\mathbb{R}^2} |f(x_1, x_2, x_3)|^{\left(1-\frac{1}{\lambda}\right)\vartheta} |\partial_2 f(x_1, x_2, x_3)| \, dx_1 dx_2 \right)^{\frac{1}{2}} \\
&\times \left(\int_{\mathbb{R}^2} |f(x_1, x_2, x_3)|^{\left(1-\frac{1}{\theta}\right)\vartheta} |\partial_3 f(x_1, x_2, x_3)| \, dx_1 dx_3 \right)^{\frac{1}{2}}.
\end{aligned}
$$

Agora, integrando com respeito a x_2 e x_3, obtemos

$$
\begin{aligned}
\int_{\mathbb{R}^3} |f|^{\vartheta} \, dx \leq C \int_{\mathbb{R}^2} &\left(\int_{\mathbb{R}} |f(x_1, x_2, x_3)|^{\left(1-\frac{1}{\lambda}\right)\vartheta} |\partial_1 f(x_1, x_2, x_3)| \, dx_1 \right)^{\frac{1}{2}} \\
&\times \left(\int_{\mathbb{R}^2} |f(x_1, x_2, x_3)|^{\left(1-\frac{1}{\lambda}\right)\vartheta} |\partial_2 f(x_1, x_2, x_3)| \, dx_1 dx_2 \right)^{\frac{1}{2}} \\
&\times \left(\int_{\mathbb{R}^2} |f(x_1, x_2, x_3)|^{\left(1-\frac{1}{\theta}\right)\vartheta} |\partial_3 f(x_1, x_2, x_3)| \, dx_1 dx_3 \right)^{\frac{1}{2}} dx_2 dx_3.
\end{aligned}
$$

Com isso,

$$
\begin{aligned}
\int_{\mathbb{R}^3} |f|^{\vartheta}\, dx \le C \int_{\mathbb{R}} & \left(\int_{\mathbb{R}^2} |f(x_1,x_2,x_3)|^{\left(1-\frac{1}{\lambda}\right)\vartheta} |\partial_2 f(x_1,x_2,x_3)|\; dx_1 dx_2 \right)^{\frac{1}{2}} \\
& \times \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x_1,x_2,x_3)|^{\left(1-\frac{1}{\lambda}\right)\vartheta} |\partial_1 f(x_1,x_2,x_3)|\; dx_1 \right)^{\frac{1}{2}} \\
& \times \left(\int_{\mathbb{R}^2} |f(x_1,x_2,x_3)|^{\left(1-\frac{1}{\theta}\right)\vartheta} |\partial_3 f(x_1,x_2,x_3)|\; dx_1 dx_3 \right)^{\frac{1}{2}} dx_2 dx_3.
\end{aligned}
$$

Consequentemente, pela Desigualdade de Hölder, infere-se que

$$
\begin{aligned}
\int_{\mathbb{R}^3} |f|^{\vartheta}\, dx \le C \left(\int_{\mathbb{R}^3} |f|^{\left(1-\frac{1}{\theta}\right)\vartheta} |\partial_3 f|\; dx \right)^{\frac{1}{2}} & \int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} |f(x_1,x_2,x_3)|^{\left(1-\frac{1}{\lambda}\right)\vartheta} |\partial_2 f(x_1,x_2,x_3)|\; dx_1 dx_2 \right)^{\frac{1}{2}} \\
\times \left(\int_{\mathbb{R}^2} |f(x_1,x_2,x_3)|^{\left(1-\frac{1}{\lambda}\right)\vartheta} |\partial_1 f(x_1,x_2,x_3)|\; dx_1 dx_2 \right)^{\frac{1}{2}} & dx_3.
\end{aligned}
$$

Deste modo, aplicando a Desigualdade de Hölder mais uma vez, chegamos a

$$
\int_{\mathbb{R}^3} |f|^{\vartheta}\, dx \le C \left(\int_{\mathbb{R}^3} |f|^{\left(1-\frac{1}{\theta}\right)\vartheta} |\partial_3 f|\; dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |f|^{\left(1-\frac{1}{\lambda}\right)\vartheta} |\partial_2 f|\; dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |f|^{\left(1-\frac{1}{\lambda}\right)\vartheta} |\partial_1 f|\; dx \right)^{\frac{1}{2}}.
$$

Assim, aplicando a Desigualdade de Hölder a cada uma das integrais encontradas do lado direito da desigualdade acima, obtemos

$$
\|f\|_{\vartheta}^{\vartheta} \le C \|f\|_{\vartheta}^{\left(1-\frac{1}{\lambda}\right)\frac{\vartheta}{2}} \|\partial_1 f\|_{\lambda}^{\frac{1}{2}} \|f\|_{\vartheta}^{\left(1-\frac{1}{\lambda}\right)\frac{\vartheta}{2}} \|\partial_2 f\|_{\lambda}^{\frac{1}{2}} \|f\|_{\vartheta}^{\left(1-\frac{1}{\theta}\right)\frac{\vartheta}{2}} \|\partial_3 f\|_{\theta}^{\frac{1}{2}}.
$$

Por conseguinte, encontramos

$$
\|f\|_{\vartheta}^{\vartheta} \le C \|f\|_{\vartheta}^{\vartheta-\frac{3}{2}} \|\partial_1 f\|_{\lambda}^{\frac{1}{2}} \|\partial_2 f\|_{\lambda}^{\frac{1}{2}} \|\partial_3 f\|_{\theta}^{\frac{1}{2}}.
$$

Por fim, obtemos

$$
\|f\|_{\vartheta}^{\frac{3}{2}} \le C \|\partial_1 f\|_{\lambda}^{\frac{1}{2}} \|\partial_2 f\|_{\lambda}^{\frac{1}{2}} \|\partial_3 f\|_{\theta}^{\frac{1}{2}}.
$$

O caso partcular segue facilmente da desigualdade acima. Isto conclui a prova do lema em questão.

□

O segundo lema também foi demonstrado em [10].

Lema 2.2 (ver [10])**.** *Seja $2 \le q \le 6$ e assuma que $f \in H^1(\mathbb{R}^3)$. Então existe uma constante* *positiva C tal que*

$$\|f\|_q \leq C\|f\|_2^{\frac{6-q}{2q}} \|\partial_1 f\|_2^{\frac{q-2}{2q}} \|\partial_2 f\|_2^{\frac{q-2}{2q}} \|\partial_3 f\|_2^{\frac{q-2}{2q}}.$$

Demonstração. A Desigualdade de Interpolação nos informa que

$$\|f\|_q \leq \|f\|_2^{\frac{6-q}{2q}} \|f\|_6^{1-\frac{6-q}{2q}} = \|f\|_2^{\frac{6-q}{2q}} \|f\|_6^{\frac{3q-6}{2q}}, \tag{2.2}$$

desde que $2 \leq q \leq 6$. Porém, por (2.1), obtemos

$$\|f\|_6 \leq C\|\partial_1 f\|_2^{\frac{1}{3}} \|\partial_2 f\|_2^{\frac{1}{3}} \|\partial_3 f\|_2^{\frac{1}{3}}.$$

Assim sendo, chegamos a

$$\|f\|_6^{\frac{3q-6}{2q}} \leq C\|\partial_1 f\|_2^{\frac{q-2}{2q}} \|\partial_2 f\|_2^{\frac{q-2}{2q}} \|\partial_3 f\|_2^{\frac{q-2}{2q}}.$$

Portanto, substituindo a desigualdade acima em (2.2), encontramos

$$\|f\|_q \leq C\|f\|_2^{\frac{6-q}{2q}} \|\partial_1 f\|_2^{\frac{q-2}{2q}} \|\partial_2 f\|_2^{\frac{q-2}{2q}} \|\partial_3 f\|_2^{\frac{q-2}{2q}}.$$

Isto completa a prova do lema em questão. □

Agora, estamos aptos a provar o resultado principal desta seção, o qual foi estabelecido em 2013 por Y. Wang [61]

Teorema 2.1 (ver [61])**.** *Seja $(\mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0) \in H^1(\mathbb{R}^3)$ tal que $\nabla \cdot \mathbf{u}_0 = \nabla \cdot \mathbf{b}_0 = 0$. Assuma que $(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t)$ é uma solução fraca para o sistema magneto-micropolar* (1) *no intervalo* $[0, T]$. *Se*

$$\int_0^T \|\partial_3 \mathbf{u}(\cdot, t)\|_p^q \, dt < \infty, \quad \frac{3}{p} + \frac{2}{q} \leq 1, \; p \geq 3, \tag{2.3}$$

então a solução $(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t)$ pode ser estendida suavemente além de $t = T$.

Demonstração. Vamos provar que $\|(\nabla\mathbf{u}, \nabla\mathbf{w}, \nabla\mathbf{b})(\cdot, t)\|_2$ é limitada por uma constante que não depende de t no intervalo de tempo $[0, T]$. Dessa forma, concluiremos que $(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t)$ pode ser estendida suavemente além de T.

Primeiramente vamos encontrar uma desigualdade de energia para a solução $(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t)$. Sendo assim, observe que se aplicarmos o produto interno $(\cdot, \mathbf{u})_2$ à primeira equação do sistema magneto-micropolar (1) obtemos

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_2^2 &= (\partial_t \mathbf{u}, \mathbf{u})_2 \\
&= (\mu+\chi)(\Delta\mathbf{u}, \mathbf{u})_2 - (\mathbf{u}\cdot\nabla\mathbf{u}, \mathbf{u})_2 + (\mathbf{b}\cdot\nabla\mathbf{b}, \mathbf{u})_2 - (\nabla(p+\frac{1}{2}|\mathbf{b}|^2), \mathbf{u})_2 + \chi(\nabla\times\mathbf{w}, \mathbf{u})_2.
\end{aligned} \tag{2.4}
$$

Agora vamos estudar cuidadosamente alguns dos termos descritos no lado direito das igualdades acima. Dessa forma, integrando por partes, chegamos a

$$
\begin{aligned}
(\Delta\mathbf{u}, \mathbf{u})_2 &= \int_{\mathbb{R}^3} \Delta\mathbf{u}\cdot\mathbf{u}\,dx = \sum_{i=1}^{3}\int_{\mathbb{R}^3} \partial_i^2\mathbf{u}\cdot\mathbf{u}\,dx \\
&= -\sum_{i=1}^{3}\int_{\mathbb{R}^3} \partial_i\mathbf{u}\cdot\partial_i\mathbf{u}\,dx = -\int_{\mathbb{R}^3}\nabla\mathbf{u}\cdot\nabla\mathbf{u}\,dx \\
&= -\|\nabla\mathbf{u}\|_2^2.
\end{aligned}
$$

Portanto, tem-se que

$$
(\Delta\mathbf{u}, \mathbf{u})_2 = -\|\nabla\mathbf{u}\|_2^2. \tag{2.5}
$$

Provaremos abaixo que também ocorre a seguinte igualdade:

$$
(\mathbf{u}\cdot\nabla\mathbf{u}, \mathbf{u})_2 = 0. \tag{2.6}
$$

Com efeito, utilizando integração por partes novamente, encontramos

$$
\begin{aligned}
\int_{\mathbb{R}^3} (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{u}\,dx &= \sum_{i=1}^{3}\int_{\mathbb{R}^3} u_i\partial_i\mathbf{u}\cdot\mathbf{u}\,dx = \sum_{i,j=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_i u_j)u_j\,dx \\
&= -\sum_{i,j=1}^{3}\int_{\mathbb{R}^3} \partial_i(u_i u_j)u_j\,dx = -\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}(\partial_i u_i)u_j^2\,dx \\
&\quad - \sum_{i,j=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_i u_j)u_j\,dx.
\end{aligned}
$$

Aqui é importante notar que

$$
\begin{aligned}
-\sum_{i,j=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_i u_j)u_j\,dx &= \sum_{i,j=1}^{3}\int_{\mathbb{R}^3} \partial_i(u_i u_j)u_j\,dx\\
&= \sum_{i,j=1}^{3}\int_{\mathbb{R}^3} (\partial_i u_i)u_j^2\,dx + \sum_{i,j=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_i u_j)u_j\,dx,
\end{aligned}
$$

isto é equivalente a

$$
-\sum_{i,j=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_i u_j)u_j\,dx = \frac{1}{2}\sum_{i,j=1}^{3}\int_{\mathbb{R}^3} (\partial_i u_i)u_j^2\,dx.
$$

Com isso, conclui-se

$$
\begin{aligned}
\int_{\mathbb{R}^3} (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{u}\,dx &= -\sum_{i,j=1}^{3}\int_{\mathbb{R}^3} (\partial_i u_i)u_j^2\,dx + \frac{1}{2}\sum_{i,j=1}^{3}\int_{\mathbb{R}^3} (\partial_i u_i)u_j^2\,dx\\
&= -\frac{1}{2}\sum_{i,j=1}^{3}\int_{\mathbb{R}^3} (\partial_i u_i)u_j^2\,dx\\
&= -\frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3} (\nabla\cdot\mathbf{u})u_j^2\,dx\\
&= 0,
\end{aligned}
$$

pois $\nabla\cdot\mathbf{u} = 0$. Agora, estamos interessados em estabelecer a igualdade abaixo:

$$
-(\nabla(p+\frac{1}{2}|\mathbf{b}|^2),\mathbf{u})_2 = 0. \tag{2.7}
$$

De fato, por integração por partes, obtemos

$$
\begin{aligned}
-(\nabla(p+\frac{1}{2}|\mathbf{b}|^2),\mathbf{u})_2 &= -\int_{\mathbb{R}^3} \nabla(p+\frac{1}{2}|\mathbf{b}|^2)\cdot\mathbf{u}\,dx\\
&= -\sum_{i=1}^{3}\int_{\mathbb{R}^3} \partial_i(p+\frac{1}{2}|\mathbf{b}|^2)u_i\,dx\\
&= \sum_{i=1}^{3}\int_{\mathbb{R}^3} (p+\frac{1}{2}|\mathbf{b}|^2)(\partial_i u_i)\,dx\\
&= \int_{\mathbb{R}^3} (p+\frac{1}{2}|\mathbf{b}|^2)(\nabla\cdot\mathbf{u})\,dx\\
&= 0,
\end{aligned}
$$

desde que $\nabla \cdot \mathbf{u} = 0$. Desta maneira, substituindo (2.5), (2.6) e (2.7) em (2.4), chega-se a

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_2^2 = -(\mu+\chi)\|\nabla\mathbf{u}\|_2^2 + (\mathbf{b}\cdot\nabla\mathbf{b},\mathbf{u})_2 + \chi(\nabla\times\mathbf{w},\mathbf{u})_2,$$

isto é,

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_2^2 + (\mu+\chi)\|\nabla\mathbf{u}\|_2^2 = \int_{\mathbb{R}^3}(\mathbf{b}\cdot\nabla\mathbf{b})\cdot\mathbf{u}\,dx + \chi\int_{\mathbb{R}^3}(\nabla\times\mathbf{w})\cdot\mathbf{u}\,dx. \tag{2.8}$$

Analogamente, aplique o produto interno $(\cdot,\mathbf{w})$ à segunda equação do sistema (1) em ordem a encontrar:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|_2^2 &= (\partial_t\mathbf{w},\mathbf{w})_2\\ &= \gamma(\Delta\mathbf{w},\mathbf{w})_2 + \kappa(\nabla(\nabla\cdot\mathbf{w}),\mathbf{w})_2 - 2\chi(\mathbf{w},\mathbf{w})_2 - (\mathbf{u}\cdot\nabla\mathbf{w},\mathbf{w})_2 + \chi(\nabla\times\mathbf{u},\mathbf{w})_2.\end{aligned} \tag{2.9}$$

Permita-nos avaliar alguns termos do lado direito das igualdades acima. Assim,

$$\begin{aligned}(\nabla(\nabla\cdot\mathbf{w}),\mathbf{w})_2 &= \int_{\mathbb{R}^3}\nabla(\nabla\cdot\mathbf{w})\cdot\mathbf{w}\,dx = \sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i(\nabla\cdot\mathbf{w})w_i\,dx\\ &= -\sum_{i=1}^{3}\int_{\mathbb{R}^3}(\nabla\cdot\mathbf{w})(\partial_i w_i)\,dx = -\int_{\mathbb{R}^3}(\nabla\cdot\mathbf{w})\,(\nabla\cdot\mathbf{w})\,dx\\ &= -\|\nabla\cdot\mathbf{w}\|_2^2.\end{aligned} \tag{2.10}$$

A seguir, vamos demonstrar que

$$-(\mathbf{u}\cdot\nabla\mathbf{w},\mathbf{w})_2 = 0. \tag{2.11}$$

De fato, é fácil ver que

$$\begin{aligned}-(\mathbf{u}\cdot\nabla\mathbf{w},\mathbf{w})_2 &= -\int_{\mathbb{R}^3}(\mathbf{u}\cdot\nabla\mathbf{w})\cdot\mathbf{w}\,dx\\ &= -\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}u_i(\partial_i w_j)w_j\,dx\\ &= \sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\partial_i(u_i w_j)w_j\,dx\\ &= \sum_{i,j=1}^{3}\int_{\mathbb{R}^3}(\partial_i u_i)w_j^2\,dx + \sum_{i,j=1}^{3}\int_{\mathbb{R}^3}u_i(\partial_i w_j)w_j\,dx.\end{aligned}$$

Por outro lado, é verdade que

$$\sum_{i,j=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_i w_j)w_j\,dx = -\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}(\partial_i u_i)w_j^2\,dx - \sum_{i,j=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_i w_j)w_j\,dx,$$

ou equivalentemente,

$$\sum_{i,j=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_i w_j)w_j\,dx = -\frac{1}{2}\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}(\partial_i u_i)w_j^2\,dx.$$

Consequentemente, chegamos a

$$\begin{aligned}
-(\mathbf{u}\cdot\nabla\mathbf{w},\mathbf{w}) &= \frac{1}{2}\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}(\partial_i u_i)w_j^2\,dx \\
&= \frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3}(\nabla\cdot\mathbf{u})w_j^2\,dx \\
&= 0,
\end{aligned}$$

pois $\mathbf{u}$ é livre de divergente. Analogamente ao que foi feito em (2.5), obtemos

$$(\Delta\mathbf{w},\mathbf{w})_2 = -\|\nabla\mathbf{w}\|_2^2. \tag{2.12}$$

Logo, por substituir os resultados encontrados em (2.10), (2.11) e (2.12) em (2.9), infere-se

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|_2^2 = -\gamma\|\nabla\mathbf{w}\|_2^2 - \kappa\|\nabla\cdot\mathbf{w}\|_2^2 - 2\chi\|\mathbf{w}\|_2^2 + \chi(\nabla\times\mathbf{u},\mathbf{w})_2,$$

ou seja,

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|_2^2 + \gamma\|\nabla\mathbf{w}\|_2^2 + \kappa\|\nabla\cdot\mathbf{w}\|_2^2 + 2\chi\|\mathbf{w}\|_2^2 = \chi\int_{\mathbb{R}^3}(\nabla\times\mathbf{u})\cdot\mathbf{w}\,dx. \tag{2.13}$$

Agora, estamos interessados em estudar a norma L^2 do campo magnético $\mathbf{b}$. Para este fim, considere o produto interno $(\cdot,\mathbf{b})_2$ à terceira equação do sistema (1) para mostrar que

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\mathbf{b}\|_2^2 &= (\partial_t\mathbf{b},\mathbf{b})_2 \\
&= \nu(\Delta\mathbf{b},\mathbf{b})_2 - (\mathbf{u}\cdot\nabla\mathbf{b},\mathbf{b})_2 + (\mathbf{b}\cdot\nabla\mathbf{u},\mathbf{b})_2.
\end{aligned} \tag{2.14}$$

Analogamente a (2.5), temos que

$$(\Delta\mathbf{b},\mathbf{b})_2 = -\|\nabla\mathbf{b}\|_2^2 \tag{2.15}$$

e, por (2.11), também temos

$$-(\mathbf{u}\cdot\nabla\mathbf{b},\mathbf{b})_2 = 0. \tag{2.16}$$

Assim sendo, substituindo (2.15) e (2.16) em (2.14), chegamos a

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{b}\|_2^2 + \nu\|\nabla\mathbf{b}\|_2^2 = \int_{\mathbb{R}^3}(\mathbf{b}\cdot\nabla\mathbf{u})\cdot\mathbf{b}\,dx. \tag{2.17}$$

Somando (2.8), (2.13) e (2.17), encontramos

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(\mathbf{u},\mathbf{w},\mathbf{b})\|_2^2 + (\mu+\chi)\|\nabla\mathbf{u}\|_2^2 + \gamma\|\nabla\mathbf{w}\|_2^2 + \nu\|\nabla\mathbf{b}\|_2^2 + \kappa\|\nabla\cdot\mathbf{w}\|_2^2 + 2\chi\|\mathbf{w}\|_2^2\\ &= \int_{\mathbb{R}^3}(\mathbf{b}\cdot\nabla\mathbf{b})\cdot\mathbf{u}\,dx + \chi\int_{\mathbb{R}^3}(\nabla\times\mathbf{w})\cdot\mathbf{u}\,dx + \chi\int_{\mathbb{R}^3}(\nabla\times\mathbf{u})\cdot\mathbf{w}\,dx + \int_{\mathbb{R}^3}(\mathbf{b}\cdot\nabla\mathbf{u})\cdot\mathbf{b}\,dx.\end{aligned} \tag{2.18}$$

Vamos agora estudar as duas parcelas que envolvem rotacionais. Sendo assim, é fácil ver que

$$(\nabla\times\mathbf{w})\cdot\mathbf{u} = (\partial_1 w_2)u_3 + (\partial_2 w_3)u_1 + (\partial_3 w_1)u_2 - (\partial_3 w_2)u_1 - (\partial_1 w_3)u_2 - (\partial_2 w_1)u_3$$

e também que

$$(\nabla\times\mathbf{u})\cdot\mathbf{w} = (\partial_1 u_2)w_3 + (\partial_2 u_3)w_1 + (\partial_3 u_1)w_2 - (\partial_3 u_2)w_1 - (\partial_1 u_3)w_2 - (\partial_2 u_1)w_3.$$

Daí, utilizando integração por partes, obtemos

$$\begin{aligned}&\chi\int_{\mathbb{R}^3}(\nabla\times\mathbf{w})\cdot\mathbf{u}\,dx + \chi\int_{\mathbb{R}^3}(\nabla\times\mathbf{u})\cdot\mathbf{w}\,dx\\ &= \chi\int_{\mathbb{R}^3}[-w_2(\partial_1 u_3) - w_3(\partial_2 u_1) - w_1(\partial_3 u_2) + w_2(\partial_3 u_1) + w_3(\partial_1 u_2) + w_1(\partial_2 u_3)]\,dx\\ &\quad + \chi\int_{\mathbb{R}^3}[(\partial_1 u_2)w_3 + (\partial_2 u_3)w_1 + (\partial_3 u_1)w_2 - (\partial_3 u_2)w_1 - (\partial_1 u_3)w_2 - (\partial_2 u_1)w_3]\,dx\\ &= 2\chi\left[\int_{\mathbb{R}^3} w_1(\partial_2 u_3 - \partial_3 u_2)\,dx\right] + 2\chi\left[\int_{\mathbb{R}^3} w_2(\partial_3 u_1 - \partial_1 u_3)\,dx\right]\\ &\quad + 2\chi\left[\int_{\mathbb{R}^3} w_3(\partial_1 u_2 - \partial_2 u_1)\,dx\right]\\ &= 2\chi\int_{\mathbb{R}^3}\mathbf{w}\cdot(\nabla\times\mathbf{u})\,dx,\end{aligned}$$

onde

$$\nabla\times\mathbf{u} = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1).$$

Consequentemente, pelas Desigualdades de Cauchy e Young, temos que

$$2\chi \int_{\mathbb{R}^3} \mathbf{w} \cdot (\nabla \times \mathbf{u})\, dx \leq 2\chi \|\mathbf{w}\|_2 \|\nabla \times \mathbf{u}\|_2 \leq \chi \|\mathbf{w}\|_2^2 + \chi \|\nabla \times \mathbf{u}\|_2^2.$$

Pela Identidade de Parseval, temos que

$$\|\nabla \times \mathbf{u}\|_2 = \|\widehat{\nabla \times \mathbf{u}}\|_2 = \|k \times \widehat{\mathbf{u}}\|_2.$$

Como

$$k \cdot \widehat{\mathbf{u}} = \sum_{j=1}^{3} k_j \widehat{u}_j = -i \sum_{j=1}^{3} i k_j \widehat{u}_j = -i \sum_{j=1}^{3} \widehat{\partial_j u_j} = -i \widehat{\nabla \cdot \mathbf{u}} = 0,$$

desde que $\nabla \cdot \mathbf{u} = 0$, então

$$\|\nabla \times \mathbf{u}\|_2^2 = \|k \times \widehat{\mathbf{u}}\|_2^2 = \int_{\mathbb{R}^3} |k|^2 |\widehat{\mathbf{u}}|^2\, dk = \int_{\mathbb{R}^3} |\widehat{\nabla \mathbf{u}}|^2\, dk = \|\nabla \mathbf{u}\|_2^2,$$

onde na última igualdade usamos novamente a Identidade de Parseval. Por fim,

$$\chi \int_{\mathbb{R}^3} (\nabla \times \mathbf{w}) \cdot \mathbf{u}\, dx + \chi \int_{\mathbb{R}^3} (\nabla \times \mathbf{u}) \cdot \mathbf{w}\, dx \leq \chi \|\mathbf{w}\|_2^2 + \chi \|\nabla \mathbf{u}\|_2^2. \tag{2.19}$$

Analisando as outras duas integrais que restam no lado direito da desigualdade (2.18), encontramos

$$\begin{aligned}
\int_{\mathbb{R}^3} (\mathbf{b} \cdot \nabla \mathbf{b}) \cdot \mathbf{u}\, dx + \int_{\mathbb{R}^3} (\mathbf{b} \cdot \nabla \mathbf{u}) \cdot \mathbf{b}\, dx &= \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} b_i (\partial_i b_j) u_j\, dx + \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} b_i (\partial_i u_j) b_j\, dx \\
&= -\sum_{i,j=1}^{3} \int_{\mathbb{R}^3} \partial_i (b_i u_j) b_j\, dx + \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} b_i (\partial_i u_j) b_j\, dx \\
&= -\sum_{i,j=1}^{3} \int_{\mathbb{R}^3} (\partial_i b_i) u_j b_j\, dx - \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} b_i (\partial_i u_j) b_j\, dx \\
&\quad + \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} b_i (\partial_i u_j) b_j\, dx \\
&= 0,
\end{aligned} \tag{2.20}$$

pois $\nabla \cdot \mathbf{b} = 0$. Portanto, substituindo (2.19) e (2.20) em (2.18), encontramos

$$\frac{1}{2}\frac{d}{dt}\|(\mathbf{u}, \mathbf{w}, \mathbf{b})\|_2^2 + \mu \|\nabla \mathbf{u}\|_2^2 + \gamma \|\nabla \mathbf{w}\|_2^2 + \nu \|\nabla \mathbf{b}\|_2^2 + \kappa \|\nabla \cdot \mathbf{w}\|_2^2 + \chi \|\mathbf{w}\|_2^2 \leq 0.$$

Integrando a desigualdade acima em relação ao intervalo $[0,t]$, temos que

$$\frac{1}{2}\|(\mathbf{u},\mathbf{w},\mathbf{b})(\cdot,t)\|_2^2 + \mu\int_0^t \|\nabla\mathbf{u}(\cdot,\tau)\|_2^2\,d\tau + \gamma\int_0^t \|\nabla\mathbf{w}(\cdot,\tau)\|_2^2\,d\tau + \nu\int_0^t \|\nabla\mathbf{b}(\cdot,\tau)\|_2^2\,d\tau$$
$$+\,\kappa\int_0^t \|\nabla\cdot\mathbf{w}(\cdot,\tau)\|_2^2\,d\tau + \chi\int_0^t \|\mathbf{w}(\cdot,\tau)\|_2^2\,d\tau + \leq \frac{1}{2}\|(\mathbf{u}_0,\mathbf{w}_0,\mathbf{b}_0)\|_2^2,$$

para todo $t\in[0,T]$.

Ainda em ordem a encontrar um limite para $\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2$, procuraremos uma estimativa para $\|(\partial_3\mathbf{u},\partial_3\mathbf{w},\partial_3\mathbf{b})\|_2$ que será útil na busca pela concretização do nosso objetivo. Dessa forma, derive a primeira equação do sistema (1), com relação a x_3, em ordem a obter

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_3\mathbf{u}\|_2^2 &= (\partial_t\partial_3\mathbf{u},\partial_3\mathbf{u})_2\\ &= (\mu+\chi)(\Delta\partial_3\mathbf{u},\partial_3\mathbf{u})_2 - (\partial_3\mathbf{u}\cdot\nabla\mathbf{u},\partial_3\mathbf{u})_2 - (\mathbf{u}\cdot\nabla\partial_3\mathbf{u},\partial_3\mathbf{u})_2 + (\partial_3\mathbf{b}\cdot\nabla\mathbf{b},\partial_3\mathbf{u})_2\\ &\quad + (\mathbf{b}\cdot\nabla\partial_3\mathbf{b},\partial_3\mathbf{u})_2 - (\nabla\partial_3(p+\frac{1}{2}|\mathbf{b}|^2),\partial_3\mathbf{u})_2 + \chi(\nabla\times\partial_3\mathbf{w},\partial_3\mathbf{u})_2. \end{aligned} \tag{2.21}$$

Permita-nos examinar alguns termos do lado direito das igualdades acima. Assim sendo, analogamente ao que foi feito em (2.6), temos que

$$\begin{aligned}(\mathbf{u}\cdot\nabla\partial_3\mathbf{u},\partial_3\mathbf{u})_2 &= \sum_{i,j=1}^3\int_{\mathbb{R}^3} u_i(\partial_i\partial_3 u_j)\partial_3 u_j\,dx\\ &= -\sum_{i,j=1}^3\int_{\mathbb{R}^3}(\partial_i u_i)(\partial_3^2 u_j)\,dx - \sum_{i,j=1}^3\int_{\mathbb{R}^3} u_i(\partial_i\partial_3 u_j)\partial_3 u_j\,dx\\ &= -\sum_{i,j=1}^3\int_{\mathbb{R}^3} u_i(\partial_i\partial_3 u_j)\partial_3 u_j\,dx,\end{aligned}$$

pois $\nabla\cdot\mathbf{u}=0$. Portanto,

$$(\mathbf{u}\cdot\nabla\partial_3\mathbf{u},\partial_3\mathbf{u})_2 = 0. \tag{2.22}$$

Observando (2.7), podemos concluir que

$$\begin{aligned}-(\nabla\partial_3(p+\frac{1}{2}|\mathbf{b}|^2),\partial_3\mathbf{u})_2 &= -\sum_{i=1}^3\int_{\mathbb{R}^3}\partial_i\partial_3(p+\frac{1}{2}|\mathbf{b}|^2)(\partial_3 u_i)\,dx\\ &= \sum_{i=1}^3\int_{\mathbb{R}^3}\partial_3(p+\frac{1}{2}|\mathbf{b}|^2)\partial_3(\partial_i u_i)\,dx\\ &= 0,\end{aligned}$$

já que $\mathbf{u}$ é livre de divergente. Logo,

$$(\nabla\partial_3(p+\frac{1}{2}|\mathbf{b}|^2),\partial_3\mathbf{u})_2=0. \tag{2.23}$$

Além disso,

$$\begin{aligned}(\Delta\partial_3\mathbf{u},\partial_3\mathbf{u})_2&=\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i^2\partial_3\mathbf{u}\cdot\partial_3\mathbf{u}\,dx\\&=-\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i\partial_3\mathbf{u}\cdot\partial_i\partial_3\mathbf{u}\,dx\\&=-\int_{\mathbb{R}^3}\nabla\partial_3\mathbf{u}\cdot\nabla\partial_3\mathbf{u}\,dx\\&=-\|\nabla\partial_3\mathbf{u}\|_2^2.\end{aligned} \tag{2.24}$$

Assim, substituindo os resultados obtidos em (2.22), (2.23) e (2.24) em (2.21), encontramos

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_3\mathbf{u}\|_2^2+(\mu+\chi)\|\nabla\partial_3\mathbf{u}\|_2^2=&-\int_{\mathbb{R}^3}(\partial_3\mathbf{u}\cdot\nabla\mathbf{u})\cdot\partial_3\mathbf{u}\,dx+\int_{\mathbb{R}^3}(\partial_3\mathbf{b}\cdot\nabla\mathbf{b})\cdot\partial_3\mathbf{u}\,dx\\&+\int_{\mathbb{R}^3}(\mathbf{b}\cdot\nabla\partial_3\mathbf{b})\cdot\partial_3\mathbf{u}\,dx+\chi\int_{\mathbb{R}^3}(\nabla\times\partial_3\mathbf{w})\cdot\partial_3\mathbf{u}\,dx.\end{aligned} \tag{2.25}$$

Derive a segunda equação do sistema (1), com relação terceira componente da variável espacial, para obter

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_3\mathbf{w}\|_2^2&=(\partial_t\partial_3\mathbf{w},\partial_3\mathbf{w})_2\\&=\gamma(\Delta\partial_3\mathbf{w},\partial_3\mathbf{w})_2+\kappa(\nabla(\nabla\cdot\partial_3\mathbf{w}),\partial_3\mathbf{w})_2-2\chi(\partial_3\mathbf{w},\partial_3\mathbf{w})_2-(\partial_3\mathbf{u}\cdot\nabla\mathbf{w},\partial_3\mathbf{w})_2\\&\quad-(\mathbf{u}\cdot\nabla\partial_3\mathbf{w},\partial_3\mathbf{w})_2+\chi(\nabla\times\partial_3\mathbf{u},\partial_3\mathbf{w})_2.\end{aligned}$$

Note que, analogamente a (2.10), é fácil obter

$$\begin{aligned}(\nabla(\nabla\cdot\partial_3\mathbf{w}),\partial_3\mathbf{w})_2&=\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i(\nabla\cdot\partial_3\mathbf{w})\partial_3 w_i\,dx\\&=-\sum_{i=1}^{3}\int_{\mathbb{R}^3}(\nabla\cdot\partial_3\mathbf{w})(\partial_3\partial_i w_i)\,dx\\&=-\int_{\mathbb{R}^3}(\nabla\cdot\partial_3\mathbf{w})\,(\nabla\cdot\partial_3\mathbf{w})\,dx\\&=-\|\nabla\cdot\partial_3\mathbf{w}\|_2^2.\end{aligned} \tag{2.26}$$

Além disso,

$$
\begin{aligned}
(\mathbf{u}\cdot\nabla\partial_3\mathbf{w},\partial_3\mathbf{w})_2 &= \sum_{i,j=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_i\partial_3 w_j)\partial_3 w_j\,dx\\
&= -\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}(\partial_i u_i)(\partial_3^2 w_j)\,dx - \sum_{i,j=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_i\partial_3 w_j)\partial_3 w_j\,dx\\
&= -\sum_{i,j=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_i\partial_3 w_j)\partial_3 w_j\,dx,
\end{aligned}
$$

já que $\mathbf{u}$ é livre de divergente. Dessa forma,

$$
(\mathbf{u}\cdot\nabla\partial_3\mathbf{w},\partial_3\mathbf{w})_2 = 0. \tag{2.27}
$$

Por utilizar (2.24), (2.26) e (2.27), conclui-se que

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_3\mathbf{w}\|_2^2+\gamma\|\nabla\partial_3\mathbf{w}\|_2^2+\kappa\|\nabla\cdot\partial_3\mathbf{w}\|_2^2+2\chi\|\partial_3\mathbf{w}\|_2^2 = -\int_{\mathbb{R}^3}(\partial_3\mathbf{u}\cdot\nabla\mathbf{w})\cdot\partial_3\mathbf{w}\,dx\\
&+\chi\int_{\mathbb{R}^3}(\nabla\times\partial_3\mathbf{u})\cdot\partial_3\mathbf{w}\,dx.
\end{aligned} \tag{2.28}
$$

E veja também que,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\partial_3\mathbf{b}\|_2^2 &= (\partial_t\partial_3\mathbf{b},\partial_3\mathbf{b})_2\\
&= \nu(\Delta\partial_3\mathbf{b},\partial_3\mathbf{b})_2-(\partial_3\mathbf{u}\cdot\nabla\mathbf{b},\partial_3\mathbf{b})_2-(\mathbf{u}\cdot\nabla\partial_3\mathbf{b},\partial_3\mathbf{b})_2\\
&\quad+(\partial_3\mathbf{b}\cdot\nabla\mathbf{u},\partial_3\mathbf{b})_2+(\mathbf{b}\cdot\nabla\partial_3\mathbf{u},\partial_3\mathbf{b})_2.
\end{aligned}
$$

Por (2.24) e (2.27), temos que

$$
(\Delta\partial_3\mathbf{b},\partial_3\mathbf{b})_2 = -\|\nabla\partial_3\mathbf{b}\|_2^2 \ \text{ e } \ (\mathbf{u}\cdot\nabla\partial_3\mathbf{b},\partial_3\mathbf{b})_2 = 0.
$$

Consequentemente,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\partial_3\mathbf{b}\|_2^2+\nu\|\nabla\partial_3\mathbf{b}\|_2^2 &= -\int_{\mathbb{R}^3}(\partial_3\mathbf{u}\cdot\nabla\mathbf{b})\cdot\partial_3\mathbf{b}\,dx+\int_{\mathbb{R}^3}(\partial_3\mathbf{b}\cdot\nabla\mathbf{u})\cdot\partial_3\mathbf{b}\,dx\\
&\quad+\int_{\mathbb{R}^3}(\mathbf{b}\cdot\nabla\partial_3\mathbf{u})\cdot\partial_3\mathbf{b}\,dx.
\end{aligned} \tag{2.29}
$$

Somando (2.25), (2.28) e (2.29), obtemos

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\partial_3\mathbf{u},\partial_3\mathbf{w},\partial_3\mathbf{b})\|_2^2+(\mu+\chi)\|\nabla\partial_3\mathbf{u}\|_2^2+\gamma\|\nabla\partial_3\mathbf{w}\|_2^2+\nu\|\nabla\partial_3\mathbf{b}\|_2^2+\kappa\|\nabla\cdot\partial_3\mathbf{w}\|_2^2\\
&+2\chi\|\partial_3\mathbf{w}\|_2^2=-\int_{\mathbb{R}^3}(\partial_3\mathbf{u}\cdot\nabla\mathbf{u})\cdot\partial_3\mathbf{u}\,dx+\int_{\mathbb{R}^3}(\partial_3\mathbf{b}\cdot\nabla\mathbf{b})\cdot\partial_3\mathbf{u}\,dx+\int_{\mathbb{R}^3}(\mathbf{b}\cdot\nabla\partial_3\mathbf{b})\cdot\partial_3\mathbf{u}\,dx\\
&+\chi\int_{\mathbb{R}^3}(\nabla\times\partial_3\mathbf{w})\cdot\partial_3\mathbf{u}\,dx-\int_{\mathbb{R}^3}(\partial_3\mathbf{u}\cdot\nabla\mathbf{w})\cdot\partial_3\mathbf{w}\,dx+\chi\int_{\mathbb{R}^3}(\nabla\times\partial_3\mathbf{u})\cdot\partial_3\mathbf{w}\,dx\\
&-\int_{\mathbb{R}^3}(\partial_3\mathbf{u}\cdot\nabla\mathbf{b})\cdot\partial_3\mathbf{b}\,dx+\int_{\mathbb{R}^3}(\partial_3\mathbf{b}\cdot\nabla\mathbf{u})\cdot\partial_3\mathbf{b}\,dx+\int_{\mathbb{R}^3}(\mathbf{b}\cdot\nabla\partial_3\mathbf{u})\cdot\partial_3\mathbf{b}\,dx.
\end{aligned}$$

Analogamente a (2.19), temos

$$\chi\int_{\mathbb{R}^3}(\nabla\times\partial_3\mathbf{w})\cdot\partial_3\mathbf{u}\,dx+\chi\int_{\mathbb{R}^3}(\nabla\times\partial_3\mathbf{u})\cdot\partial_3\mathbf{w}\,dx\leq\chi\|\partial_3\mathbf{w}\|_2^2+\chi\|\nabla\partial_3\mathbf{u}\|_2^2, \tag{2.30}$$

já que $\nabla\cdot\mathbf{u}=0$. Da mesma forma que em (2.20), temos

$$\begin{aligned}
&\int_{\mathbb{R}^3}(\mathbf{b}\cdot\nabla\partial_3\mathbf{b})\cdot\partial_3\mathbf{u}\,dx+\int_{\mathbb{R}^3}(\mathbf{b}\cdot\nabla\partial_3\mathbf{u})\cdot\partial_3\mathbf{b}\,dx=-\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}b_i(\partial_i\partial_3u_j)(\partial_3b_j)\,dx\\
&+\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}b_i(\partial_3\partial_iu_j)(\partial_3b_j)\,dx=0,
\end{aligned}\tag{2.31}$$

pois $\mathbf{b}$ é livre de divergente. Assim, por (2.30) e (2.31), obtemos

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\partial_3\mathbf{u},\partial_3\mathbf{w},\partial_3\mathbf{b})\|_2^2+\mu\|\nabla\partial_3\mathbf{u}\|_2^2+\gamma\|\nabla\partial_3\mathbf{w}\|_2^2+\nu\|\nabla\partial_3\mathbf{b}\|_2^2+\kappa\|\nabla\cdot\partial_3\mathbf{w}\|_2^2+\chi\|\partial_3\mathbf{w}\|_2^2\\
&\leq-\int_{\mathbb{R}^3}(\partial_3\mathbf{u}\cdot\nabla\mathbf{u})\cdot\partial_3\mathbf{u}\,dx+\int_{\mathbb{R}^3}(\partial_3\mathbf{b}\cdot\nabla\mathbf{b})\cdot\partial_3\mathbf{u}\,dx-\int_{\mathbb{R}^3}(\partial_3\mathbf{u}\cdot\nabla\mathbf{w})\cdot\partial_3\mathbf{w}\,dx\\
&-\int_{\mathbb{R}^3}(\partial_3\mathbf{u}\cdot\nabla\mathbf{b})\cdot\partial_3\mathbf{b}\,dx+\int_{\mathbb{R}^3}(\partial_3\mathbf{b}\cdot\nabla\mathbf{u})\cdot\partial_3\mathbf{b}\,dx=:I_1+I_2+I_3+I_4+I_5.
\end{aligned}\tag{2.32}$$

No que segue, iremos estimar I_j $(j=1,...,5)$. Por integração por partes, temos que

$$\begin{aligned}
I_1&:=-\int_{\mathbb{R}^3}(\partial_3\mathbf{u}\cdot\nabla\mathbf{u})\cdot\partial_3\mathbf{u}\,dx\\
&=-\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}(\partial_3u_i)(\partial_iu_j)(\partial_3u_j)\,dx\\
&=\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}(\partial_i\partial_3u_j)(\partial_3u_i)u_j\,dx+\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}(\partial_3u_j)(\partial_i\partial_3u_i)u_j\,dx\\
&=\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}(\partial_i\partial_3u_j)(\partial_3u_i)u_j\,dx,
\end{aligned}$$

pois $\nabla \cdot \mathbf{u} = 0$. Portanto, pela Desigualdade de Hölder, obtemos

$$I_1 \leq \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} |\partial_i \partial_3 u_j||\partial_3 u_i||u_j|\, dx \leq C \int_{\mathbb{R}^3} |\nabla \partial_3 \mathbf{u}||\partial_3 \mathbf{u}||\mathbf{u}|\, dx \leq C\|\nabla \partial_3 \mathbf{u}\|_2 \|\partial_3 \mathbf{u}\|_\rho \|\|\mathbf{u}\|_{3p},$$

onde

$$\frac{1}{\rho} + \frac{1}{3p} = \frac{1}{2}, \quad 2 \leq \rho \leq 6.$$

Segue da Desigualdade de Gagliardo-Nirenberg (1.2), do Lema 2.1 e da Desigualdade de Young, que

$$\begin{aligned}
I_1 \leq\ & C\|\nabla \partial_3 \mathbf{u}\|_2 \|\partial_3 \mathbf{u}\|_2^{1-3(\frac{1}{2}-\frac{1}{\rho})} \|\nabla \partial_3 \mathbf{u}\|_2^{3(\frac{1}{2}-\frac{1}{\rho})} \|\mathbf{u}\|_{3p} \\
\leq\ & C\|\nabla \partial_3 \mathbf{u}\|_2 \|\partial_3 \mathbf{u}\|_2^{1-3(\frac{1}{2}-\frac{1}{\rho})} \|\nabla \partial_3 \mathbf{u}\|_2^{3(\frac{1}{2}-\frac{1}{\rho})} \|\partial_1 \mathbf{u}\|_2^{\frac{1}{3}} \|\partial_2 \mathbf{u}\|_2^{\frac{1}{3}} \|\partial_3 \mathbf{u}\|_p^{\frac{1}{3}} \\
\leq\ & C\|\nabla \partial_3 \mathbf{u}\|_2^{1+3(\frac{1}{2}-\frac{1}{\rho})} \|\partial_3 \mathbf{u}\|_2^{1-3(\frac{1}{2}-\frac{1}{\rho})} \|\nabla \mathbf{u}\|_2^{\frac{2}{3}} \|\partial_3 \mathbf{u}\|_p^{\frac{1}{3}} \\
\leq\ & \frac{\mu}{2}\|\nabla \partial_3 \mathbf{u}\|_2^2 + C\|\partial_3 \mathbf{u}\|_2^2 \|\nabla \mathbf{u}\|_2^{2r} \|\partial_3 \mathbf{u}\|_p^r,
\end{aligned} \tag{2.33}$$

onde $r = \frac{2p}{3(p-1)}$. Quando $p > 3$, temos $r < 1$. Daí, usando a Desigualdade de Young novamente, encontramos

$$\begin{aligned}
I_1 \leq\ & \frac{\mu}{2}\|\nabla \partial_3 \mathbf{u}\|_2^2 + C\|\partial_3 \mathbf{u}\|_2^2 (r\|\nabla \mathbf{u}\|_2^2 + (1-r)\|\partial_3 \mathbf{u}\|_p^{\frac{r}{1-r}}) \\
\leq\ & \frac{\mu}{2}\|\nabla \partial_3 \mathbf{u}\|_2^2 + C\|\partial_3 \mathbf{u}\|_2^2 (\|\nabla \mathbf{u}\|_2^2 + \|\partial_3 \mathbf{u}\|_p^{\delta}),
\end{aligned} \tag{2.34}$$

onde

$$\delta = \frac{r}{1-r} = \frac{2p}{p-3} \text{ e } \frac{3}{p} + \frac{2}{\delta} = 1.$$

Agora, vamos encontrar uma estimativa para I_2. Primeiramente, note que, pela Desigualdade de Hölder, temos

$$\begin{aligned}
I_2 := \int_{\mathbb{R}^3} (\partial_3 \mathbf{b} \cdot \nabla \mathbf{b}) \cdot \partial_3 \mathbf{u}\, dx &= \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} (\partial_3 b_i)(\partial_i b_j)(\partial_3 u_j)\, dx \\
&\leq C \int_{\mathbb{R}^3} |\partial_3 \mathbf{b}||\nabla \mathbf{b}||\partial_3 \mathbf{u}|\, dx \leq C\|\nabla \mathbf{b}\|_2 \|\partial_3 \mathbf{b}\|_{\frac{2p}{p-2}} \|\partial_3 \mathbf{u}\|_p.
\end{aligned}$$

Usando as Desigualdades de Gagliardo-Nirenberg (1.2) e Young, obtemos

$$\begin{aligned}
I_2 \leq\ & C\|\nabla \mathbf{b}\|_2\ \|\partial_3 \mathbf{u}\|_p\ \|\partial_3 \mathbf{b}\|_2^{1-\frac{3}{p}}\ \|\nabla \partial_3 \mathbf{b}\|_2^{\frac{3}{p}} \\
\leq\ & \frac{\nu}{6}\|\nabla \partial_3 \mathbf{b}\|_2^2 + C\|\nabla \mathbf{b}\|_2^{\frac{2p}{2p-3}}\ \|\partial_3 \mathbf{u}\|_p^{\frac{2p}{2p-3}}\ \|\partial_3 \mathbf{b}\|_2^{\frac{2p-6}{2p-3}}.
\end{aligned}$$

Novamente, pela Desigualdade de Young, temos que

$$I_2 \leq \frac{\nu}{6}\|\nabla\partial_3\mathbf{b}\|_2^2 + C(\|\nabla\mathbf{b}\|_2^2 + \|\partial_3\mathbf{u}\|_p^{\delta})\ \|\partial_3\mathbf{b}\|_2^{\frac{2p-6}{2p-3}}. \tag{2.35}$$

Analogamente ao que foi feito para I_2 podemos estimar I_3. De fato, pela Desigualdade de Hölder, concluímos que

$$\begin{aligned} I_3 &= -\int_{\mathbb{R}^3} (\partial_3\mathbf{u}\cdot\nabla\mathbf{w})\cdot\partial_3\mathbf{w}\,dx = -\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}(\partial_3 u_i)(\partial_i w_j)(\partial_3 w_j)\,dx \\ &\leq\ C\int_{\mathbb{R}^3}|\partial_3\mathbf{u}||\nabla\mathbf{w}||\partial_3\mathbf{w}|\,dx \leq C\|\nabla\mathbf{w}\|_2\|\partial_3\mathbf{w}\|_{\frac{2p}{p-2}}\|\partial_3\mathbf{u}\|_p. \end{aligned}$$

Novamente pelas Desigualdades (1.2) e Young, chegamos a

$$\begin{aligned} I_3 &\leq C\|\nabla\mathbf{w}\|_2\|\partial_3\mathbf{u}\|_p\|\partial_3\mathbf{w}\|_2^{1-\frac{3}{p}}\|\nabla\partial_3\mathbf{w}\|_2^{\frac{3}{p}} \\ &\leq \frac{\gamma}{2}\|\nabla\partial_3\mathbf{w}\|_2^2 + C\|\nabla\mathbf{w}\|_2^{\frac{2p}{2p-3}}\|\partial_3\mathbf{u}\|_p^{\frac{2p}{2p-3}}\|\partial_3\mathbf{w}\|_2^{\frac{2p-6}{2p-3}} \\ &\leq \frac{\gamma}{2}\|\nabla\partial_3\mathbf{w}\|_2^2 + C(\|\nabla\mathbf{w}\|_2^2 + \|\partial_3\mathbf{u}\|_p^{\delta})\|\partial_3\mathbf{w}\|_2^{\frac{2p-6}{2p-3}}. \end{aligned} \tag{2.36}$$

Agora, estamos interessados em encontrar um limite para I_4. Sendo assim,

$$\begin{aligned} I_4 &:= -\int_{\mathbb{R}^3}(\partial_3\mathbf{u}\cdot\nabla\mathbf{b})\cdot\partial_3\mathbf{b}\,dx \\ &= -\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}(\partial_3 u_i)(\partial_i b_j)(\partial_3 b_j)\,dx \\ &\leq C\int_{\mathbb{R}^3}|\partial_3\mathbf{u}||\nabla\mathbf{b}||\partial_3\mathbf{b}|\,dx \\ &\leq C\|\nabla\mathbf{b}\|_2\|\partial_3\mathbf{b}\|_{\frac{2p}{p-2}}\|\partial_3\mathbf{u}\|_p \\ &\leq C\|\nabla\mathbf{b}\|_2\|\partial_3\mathbf{u}\|_p\|\partial_3\mathbf{b}\|_2^{1-\frac{3}{p}}\|\nabla\partial_3\mathbf{b}\|_2^{\frac{3}{p}} \\ &\leq \frac{\nu}{6}\|\nabla\partial_3\mathbf{b}\|_2^2 + C\|\nabla\mathbf{b}\|_2^{\frac{2p}{2p-3}}\|\partial_3\mathbf{u}\|_p^{\frac{2p}{2p-3}}\|\partial_3\mathbf{b}\|_2^{\frac{2p-6}{2p-3}} \\ &\leq\ \frac{\nu}{6}\|\nabla\partial_3\mathbf{b}\|_2^2 + C(\|\nabla\mathbf{b}\|_2^2 + \|\partial_3\mathbf{u}\|_p^{\delta})\|\partial_3\mathbf{b}\|_2^{\frac{2p-6}{2p-3}}. \end{aligned} \tag{2.37}$$

Por fim, estimemos I_5. É fácil ver que

$$\begin{aligned}
I_5 &:= \int_{\mathbb{R}^3} (\partial_3\mathbf{b}\cdot\nabla\mathbf{u})\cdot\partial_3\mathbf{b}\,dx\\
&= \sum_{i,j=1}^{3}\int_{\mathbb{R}^3} (\partial_3 b_i)(\partial_i u_j)(\partial_3 b_j)\,dx\\
&= -\sum_{i,j=1}^{3}\int_{\mathbb{R}^3} (\partial_i\partial_3 b_j)(\partial_3 b_i)u_j\,dx - \sum_{i,j=1}^{3}\int_{\mathbb{R}^3} (\partial_3 b_j)(\partial_i\partial_3 b_i)u_j\,dx\\
&= -\sum_{i,j=1}^{3}\int_{\mathbb{R}^3} (\partial_i\partial_3 b_j)(\partial_3 b_i)u_j\,dx,
\end{aligned}$$

pois $\nabla\cdot\mathbf{b}=0$. Deste modo, pela Desigualdade de Hölder, infere-se

$$I_5 \le \sum_{i,j=1}^{3}\int_{\mathbb{R}^3} |\partial_i\partial_3 b_j|\ |\partial_3 b_i|\ |u_j|\,dx \le C\int_{\mathbb{R}^3} |\nabla\partial_3\mathbf{b}||\partial_3\mathbf{b}||\mathbf{u}|\,dx \le C\|\nabla\partial_3\mathbf{b}\|_2\|\partial_3\mathbf{b}\|_\rho\|\mathbf{u}\|_{3p}.$$

Pelas Desigualdades de Gagliardo-Nirenberg e Young, e também pelo Lema 2.1 chegamos a

$$\begin{aligned}
I_5 \le&\ C\|\nabla\partial_3\mathbf{b}\|_2\ \|\partial_3\mathbf{b}\|_2^{1-3\left(\frac{1}{2}-\frac{1}{\rho}\right)}\ \|\nabla\partial_3\mathbf{b}\|_2^{3\left(\frac{1}{2}-\frac{1}{\rho}\right)}\|\mathbf{u}\|_{3p}\\
\le&\ C\|\nabla\partial_3\mathbf{b}\|_2\ \|\partial_3\mathbf{b}\|_2^{1-3\left(\frac{1}{2}-\frac{1}{\rho}\right)}\ \|\nabla\partial_3\mathbf{b}\|_2^{3\left(\frac{1}{2}-\frac{1}{\rho}\right)}\ \|\partial_1\mathbf{u}\|_2^{\frac{1}{3}}\|\partial_2\mathbf{u}\|_2^{\frac{1}{3}}\|\partial_3\mathbf{u}\|_p^{\frac{1}{3}}\\
\le&\ C\|\nabla\partial_3\mathbf{b}\|_2\ \|\partial_3\mathbf{b}\|_2^{1-3\left(\frac{1}{2}-\frac{1}{\rho}\right)}\ \|\nabla\partial_3\mathbf{b}\|_2^{3\left(\frac{1}{2}-\frac{1}{\rho}\right)}\ \|\nabla\mathbf{u}\|_2^{\frac{2}{3}}\ \|\partial_3\mathbf{u}\|_p^{\frac{1}{3}}\\
\le&\ \frac{\nu}{6}\|\nabla\partial_3\mathbf{b}\|_2^2 + C\|\partial_3\mathbf{b}\|_2^2\ \|\nabla\mathbf{u}\|_2^{2r}\ \|\partial_3\mathbf{u}\|_p^r,
\end{aligned}$$

onde $r=\frac{2p}{3(p-1)}$. Quando $p>3$, temos que $r<1$. Daí, usando Desigualdade de Young, obtemos

$$I_5 \le\ \frac{\nu}{6}\|\nabla\partial_3\mathbf{b}\|_2^2 + C\|\partial_3\mathbf{b}\|_2^2(\|\nabla\mathbf{u}\|_2^2+\|\partial_3\mathbf{u}\|_p^\delta), \tag{2.38}$$

onde $\delta=\frac{2p}{p-3}$ e $\frac{3}{p}+\frac{2}{\delta}=1$. Combinando (2.32)-(2.38), encontramos

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\partial_3\mathbf{u},\partial_3\mathbf{w},\partial_3\mathbf{b})\|_2^2 + \frac{\mu}{2}\|\nabla\partial_3\mathbf{u}\|_2^2 + \frac{\gamma}{2}\|\nabla\partial_3\mathbf{w}\|_2^2 + \frac{\nu}{2}\|\nabla\partial_3\mathbf{b}\|_2^2 + \kappa\|\nabla\cdot\partial_3\mathbf{w}\|_2^2 + \chi\|\partial_3\mathbf{w}\|_2^2\\
&\le\ C\|\partial_3\mathbf{u}\|_2^2(\|\nabla\mathbf{u}\|_2^2+\|\partial_3\mathbf{u}\|_p^\delta) + C(\|\nabla\mathbf{b}\|_2^2+\|\partial_3\mathbf{u}\|_p^\delta)\|\partial_3\mathbf{b}\|_2^{\frac{2p-6}{2p-3}} + C(\|\nabla\mathbf{w}\|_2^2+\|\partial_3\mathbf{u}\|_p^\delta)\\
&\times\ \|\partial_3\mathbf{w}\|_2^{\frac{2p-6}{2p-3}} + C\|\partial_3\mathbf{b}\|_2^2(\|\nabla\mathbf{u}\|_2^2+\|\partial_3\mathbf{u}\|_p^\delta).
\end{aligned}$$

Consequentemente, pela Desigualdade de Young, obtemos

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\partial_3\mathbf{u},\partial_3\mathbf{w},\partial_3\mathbf{b})\|_2^2+\frac{\mu}{2}\|\nabla\partial_3\mathbf{u}\|_2^2+\frac{\gamma}{2}\|\nabla\partial_3\mathbf{w}\|_2^2+\frac{\nu}{2}\|\nabla\partial_3\mathbf{b}\|_2^2+\kappa\|\nabla\cdot\partial_3\mathbf{w}\|_2^2+\chi\|\partial_3\mathbf{w}\|_2^2\\
\leq\ & C(\|\nabla\mathbf{u}\|_2^2+\|\nabla\mathbf{w}\|_2^2+\|\nabla\mathbf{b}\|_2^2+\|\partial_3\mathbf{u}\|_p^\delta)(\|\partial_3\mathbf{u}\|_2^2+\|\partial_3\mathbf{w}\|_2^2+\|\partial_3\mathbf{b}\|_2^2+\|\partial_3\mathbf{u}\|_2^{\frac{2p-6}{2p-3}}\\
&+\|\partial_3\mathbf{w}\|_2^{\frac{2p-6}{2p-3}}+\|\partial_3\mathbf{b}\|_2^{\frac{2p-6}{2p-3}})\\
\leq\ & C(\|\nabla\mathbf{u}\|_2^2+\|\nabla\mathbf{w}\|_2^2+\|\nabla\mathbf{b}\|_2^2+\|\partial_3\mathbf{u}\|_p^\delta)(\|\partial_3\mathbf{u}\|_2^2+\|\partial_3\mathbf{w}\|_2^2+\|\partial_3\mathbf{b}\|_2^2+C\cdot 1^{\frac{2p-3}{p}}\\
&+C(\|\partial_3\mathbf{u}\|_2^{\frac{2p-6}{2p-3}}+\|\partial_3\mathbf{w}\|_2^{\frac{2p-6}{2p-3}}+\|\partial_3\mathbf{b}\|_2^{\frac{2p-6}{2p-3}})^{\frac{2p-3}{p-3}})\\
\leq\ & C(\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^2+\|\partial_3\mathbf{u}\|_p^\delta)(1+\|(\partial_3\mathbf{u},\partial_3\mathbf{w},\partial_3\mathbf{b})\|_2^2),
\end{aligned}
\tag{2.39}
$$

onde no último passo utilizamos a desigualdade $(a+b+c)^p\leq C(a^p+b^p+c^p)$. Por aplicar o Lema de Gronwall, a desigualdade (1.1) e a hipótese (2.3), segue que

$$
\begin{aligned}
\|(\partial_3\mathbf{u},\partial_3\mathbf{w},\partial_3\mathbf{b})(\cdot,t)\|_2^2\leq\ & e^{C\int_0^t[\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\cdot,\tau)\|_2^2+\|\partial_3\mathbf{u}(\cdot,\tau)\|_p^\delta]\,d\tau}[\|(\partial_3\mathbf{u}_0,\partial_3\mathbf{w}_0,\partial_3\mathbf{b}_0)\|_2^2\\
&+C\int_0^t(\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\cdot,\tau)\|_2^2+\|\partial_3\mathbf{u}(\cdot,\tau)\|_p^\delta)\,d\tau]\\
\leq\ & e^{C(\|(\mathbf{u_0},\mathbf{w_0},\mathbf{b_0})\|_2^2+\int_0^T\|\partial_3\mathbf{u}(\cdot,t)\|_p^\delta\,dt)}[\|(\partial_3\mathbf{u}_0,\partial_3\mathbf{w}_0,\partial_3\mathbf{b}_0)\|_2^2\\
&+C\|(\mathbf{u}_0,\mathbf{w}_0,\mathbf{b}_0)\|_2^2+\int_0^T\|\partial_3\mathbf{u}(\cdot,t)\|_p^\delta\,dt]\\
\leq\ & C.
\end{aligned}
\tag{2.40}
$$

Integrando (2.39) em relação ao intervalo $[0,t]$, com $0\leq t\leq T$ e por usar (2.40), (1.1) e (2.3), obtemos

$$
\begin{aligned}
&\|(\partial_3\mathbf{u},\partial_3\mathbf{w},\partial_3\mathbf{b})(\cdot,t)\|_2^2+\mu\int_0^t\|\nabla\partial_3\mathbf{u}(\cdot,s)\|_2^2\,ds+\gamma\int_0^t\|\nabla\partial_3\mathbf{w}(\cdot,s)\|_2^2\,ds+\nu\int_0^t\|\nabla\partial_3\mathbf{b}(\cdot,s)\|_2^2\,ds\\
&+2\kappa\int_0^t\|\nabla\cdot\partial_3\mathbf{w}(\cdot,s)\|_2^2\,ds+2\chi\int_0^t\|\nabla\partial_3\mathbf{w}(\cdot,s)\|_2^2\,ds\leq C\int_0^t\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\cdot,s)\|_2^2\,ds\\
&+C\int_0^t\|\partial_3\mathbf{u}(\cdot,s)\|_p^\delta\,ds\leq\|(\partial_3\mathbf{u}_0,\partial_3\mathbf{w}_0,\partial_3\mathbf{b}_0)\|_2^2+C(\|(\mathbf{u}_0,\mathbf{w}_0,\mathbf{b}_0)\|_2^2+\int_0^T\|\partial_3\mathbf{u}(\cdot,t)\|_p^\delta\,dt)\leq C,
\end{aligned}
\tag{2.41}
$$

para todo $t\in[0,T]$.

Considerando $p=3$, temos $r=1$. Assim, busquemos estimativas para I_j, com $j=1,\ldots,5$.

Note que,

$$
\begin{aligned}
I_1 &\leq \frac{\mu}{2}\|\nabla\partial_3\mathbf{u}\|_2^2 + C\|\partial_3\mathbf{u}\|_2^2\|\nabla\mathbf{u}\|_2^2\|\partial_3\mathbf{u}\|_3 \\
&\leq \frac{\mu}{2}\|\nabla\partial_3\mathbf{u}\|_2^2 + C\|\partial_3\mathbf{u}\|_2^2\|\nabla\mathbf{u}\|_2^2,
\end{aligned} \tag{2.42}
$$

onde nesta última desigualdade usamos a hipótese. Agora, observe que pelo Lema 2.1, por hipótese e pela Desigualdade de Young, temos

$$
\begin{aligned}
I_2 &\leq C\|\nabla\mathbf{b}\|_2\|\partial_3\mathbf{b}\|_6\|\partial_3\mathbf{u}\|_3 \\
&\leq C\|\nabla\mathbf{b}\|_2\|\partial_1\partial_3\mathbf{b}\|_2^{\frac{1}{3}}\|\partial_2\partial_3\mathbf{b}\|_2^{\frac{1}{3}}\|\partial_3\partial_3\mathbf{b}\|_2^{\frac{1}{3}} \\
&\leq C\|\nabla\mathbf{b}\|_2\|\nabla\partial_3\mathbf{b}\|_2 \\
&\leq \frac{\nu}{6}\|\nabla\partial_3\mathbf{b}\|_2^2 + C\|\nabla\mathbf{b}\|_2^2.
\end{aligned} \tag{2.43}
$$

Analogamente ao feito para I_2 podemos estimar I_3. Com efeito,

$$
\begin{aligned}
I_3 &\leq C\|\nabla\mathbf{w}\|_2\|\partial_3\mathbf{w}\|_6\|\partial_3\mathbf{u}\|_3 \\
&\leq C\|\nabla\mathbf{w}\|_2\|\partial_1\partial_3\mathbf{w}\|_2^{\frac{1}{3}}\|\partial_2\partial_3\mathbf{w}\|_2^{\frac{1}{3}}\|\partial_3\partial_3\mathbf{w}\|_2^{\frac{1}{3}} \\
&\leq C\|\nabla\mathbf{w}\|_2\|\nabla\partial_3\mathbf{w}\|_2 \\
&\leq \frac{\gamma}{2}\|\nabla\partial_3\mathbf{w}\|_2^2 + C\|\nabla\mathbf{w}\|_2^2.
\end{aligned} \tag{2.44}
$$

Agora, estimando I_4, obtemos

$$
\begin{aligned}
I_4 &\leq C\|\nabla\mathbf{b}\|_2\|\partial_3\mathbf{b}\|_6\|\partial_3\mathbf{u}\|_3 \\
&\leq C\|\nabla\mathbf{b}\|_2\|\partial_1\partial_3\mathbf{b}\|_2^{\frac{1}{3}}\|\partial_2\partial_3\mathbf{b}\|_2^{\frac{1}{3}}\|\partial_3\partial_3\mathbf{b}\|_2^{\frac{1}{3}} \\
&\leq C\|\nabla\mathbf{b}\|_2\|\nabla\partial_3\mathbf{b}\|_2 \\
&\leq \frac{\nu}{6}\|\nabla\partial_3\mathbf{b}\|_2^2 + C\|\nabla\mathbf{b}\|_2^2.
\end{aligned} \tag{2.45}
$$

Por fim, estimemos I_5. Assim,

$$
\begin{aligned}
I_5 &\leq \frac{\nu}{6}\|\nabla\partial_3\mathbf{b}\|_2^2 + C\|\partial_3\mathbf{b}\|_2^2\|\nabla\mathbf{u}\|_2^2\|\partial_3\mathbf{u}\|_3 \\
&\leq \frac{\nu}{6}\|\nabla\partial_3\mathbf{b}\|_2^2 + C\|\partial_3\mathbf{b}\|_2^2\|\nabla\mathbf{u}\|_2^2.
\end{aligned} \tag{2.46}
$$

Combinando (2.42)-(2.46), encontramos

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\partial_3\mathbf{u},\partial_3\mathbf{w},\partial_3\mathbf{b})\|_2^2+\frac{\mu}{2}\|\nabla\partial_3\mathbf{u}\|_2^2+\frac{\gamma}{2}\|\nabla\partial_3\mathbf{w}\|_2^2+\frac{\nu}{2}\|\nabla\partial_3\mathbf{b}\|_2^2+\kappa\|\nabla\cdot\partial_3\mathbf{w}\|_2^2+\chi\|\partial_3\mathbf{w}\|_2^2\\
&\leq C\|\partial_3\mathbf{u}\|_2^2\|\nabla\mathbf{u}\|_2^2+C\|\nabla\mathbf{b}\|_2^2+C\|\nabla\mathbf{w}\|_2^2+C\|\partial_3\mathbf{b}\|_2^2\|\nabla\mathbf{u}\|_2^2\\
&\leq C\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^2+C\|(\partial_3\mathbf{u},\partial_3\mathbf{w},\partial_3\mathbf{b})\|_2^2\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^2\\
&\leq C\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^2(1+\|(\partial_3\mathbf{u},\partial_3\mathbf{w},\partial_3\mathbf{b})\|_2^2). \qquad (2.47)
\end{aligned}$$

Aplicando o Lema de Gronwall e (1.1), obtemos

$$\begin{aligned}
\|(\partial_3\mathbf{u},\partial_3\mathbf{w},\partial_3\mathbf{b})(\cdot,t)\|_2^2 &\leq e^{C\int_0^t\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\cdot,\tau)\|_2^2\,d\tau}[\|(\partial_3\mathbf{u}_0,\partial_3\mathbf{w}_0,\partial_3\mathbf{b}_0)\|_2^2\\
&\quad+\int_0^t\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\cdot,\tau)\|_2^2\,d\tau]\\
&\leq e^{C\int_0^T\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\cdot,\tau)\|_2^2\,d\tau}[\|(\partial_3\mathbf{u}_0,\partial_3\mathbf{w}_0,\partial_3\mathbf{b}_0)\|_2^2\\
&\quad+\int_0^T\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\cdot,\tau)\|_2^2\,d\tau]\\
&\leq C. \qquad (2.48)
\end{aligned}$$

Integrando (2.47) em relação ao intervalo $[0,t]$, com $0\leq t\leq T$ e utilizando (2.48) e (1.1), temos

$$\begin{aligned}
&\|(\partial_3\mathbf{u},\partial_3\mathbf{w},\partial_3\mathbf{b})(\cdot,t)\|_2^2+\mu\int_0^t\|\nabla\partial_3\mathbf{u}(\cdot,s)\|_2^2\,ds+\gamma\int_0^t\|\nabla\partial_3\mathbf{w}(\cdot,s)\|_2^2\,ds+\nu\int_0^t\|\nabla\partial_3\mathbf{b}(\cdot,s)\|_2^2\,ds\\
&+2\kappa\int_0^t\|\nabla\cdot\partial_3\mathbf{w}(\cdot,s)\|_2^2\,ds+2\chi\int_0^t\|\partial_3\mathbf{w}(\cdot,s)\|_2^2\,ds\leq C\int_0^t\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\cdot,s)\|_2^2\\
&\times(1+\|(\partial_3\mathbf{u},\partial_3\mathbf{w},\partial_3\mathbf{b})(\cdot,s)\|_2^2)\,ds\leq C,\quad\forall\,t\in[0,T]. \qquad (2.49)
\end{aligned}$$

Estamos prontos para provar que a norma-L^2 do gradiente da solução $(\mathbf{u},\mathbf{w},\mathbf{b})(\cdot,t)$ de (1) é limitada em $[0,T]$. Primeiramente, se aplicarmos $(\cdot,\Delta\mathbf{u})_2$ à primeira equação do sistema (1), obtemos

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|_2^2 &= (\nabla(\partial_t\mathbf{u}),\nabla\mathbf{u})_2=\sum_{i=1}^{3}(\partial_i\partial_t\mathbf{u},\partial_i\mathbf{u})_2\\
&=-\sum_{i=1}^{3}(\partial_t\mathbf{u},\partial_i^2\mathbf{u})_2=-(\partial_t\mathbf{u},\Delta\mathbf{u})_2\\
&=-(\mu+\chi)(\Delta\mathbf{u},\Delta\mathbf{u})_2+(\mathbf{u}\cdot\nabla\mathbf{u},\Delta\mathbf{u})_2-(\mathbf{b}\cdot\nabla\mathbf{b},\Delta\mathbf{u})_2+(\nabla(p+\frac{1}{2}|\mathbf{b}|^2),\Delta\mathbf{u})_2\\
&\quad-\chi(\nabla\times\mathbf{w},\Delta\mathbf{u})_2.
\end{aligned}$$

Vamos mostrar que a parcela que envolve a pressão é nula. De fato,

$$
\begin{aligned}
(\nabla(p+\frac{1}{2}|\mathbf{b}|^2), \Delta\mathbf{u})_2 &= \int_{\mathbb{R}^3} \nabla(p+\frac{1}{2}|\mathbf{b}|^2)\cdot\Delta\mathbf{u}\,dx \\
&= \sum_{i=1}^{3}\int_{\mathbb{R}^3} \partial_i(p+\frac{1}{2}|\mathbf{b}|^2)\ \Delta u_i\,dx \\
&= -\sum_{i=1}^{3}\int_{\mathbb{R}^3} (p+\frac{1}{2}|\mathbf{b}|^2)\ \Delta(\partial_i u_i)\,dx \\
&= 0,
\end{aligned}
$$

pois $\nabla\cdot\mathbf{u}=0$. Assim,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|_2^2 + (\mu+\chi)\|\Delta\mathbf{u}\|_2^2 &= \int_{\mathbb{R}^3}(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\Delta\mathbf{u}\,dx - \int_{\mathbb{R}^3}(\mathbf{b}\cdot\nabla\mathbf{b})\cdot\Delta\mathbf{u}\,dx \\
&\quad - \chi\int_{\mathbb{R}^3}(\nabla\times\mathbf{w})\cdot\Delta\mathbf{u}\,dx.
\end{aligned} \tag{2.50}
$$

Analogamente ao que foi feito para a primeira equação de (1), podemos inferir o seguinte sobre a segunda equação deste mesmo sistema:

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{w}\|_2^2 &= -(\partial_t\mathbf{w},\Delta\mathbf{w})_2 \\
&= -\gamma\|\Delta\mathbf{w}\|_2^2 - \kappa(\nabla(\nabla\cdot\mathbf{w}),\Delta\mathbf{w})_2 + 2\chi(\mathbf{w},\Delta\mathbf{w})_2 + (\mathbf{u}\cdot\nabla\mathbf{w},\Delta\mathbf{w})_2 - \chi(\nabla\times\mathbf{u},\Delta\mathbf{w})_2.
\end{aligned}
$$

É fácil ver que

$$
\begin{aligned}
(\nabla(\nabla\cdot\mathbf{w}),\Delta\mathbf{w})_2 &= -\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\partial_i(\nabla\cdot\mathbf{w})(\partial_j^2 w_i)\,dx \\
&= \sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\partial_j\partial_i(\nabla\cdot\mathbf{w})(\partial_j w_i)\,dx \\
&= -\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\partial_j(\nabla\cdot\mathbf{w})(\partial_j\partial_i w_i)\,dx \\
&= -\sum_{j=1}^{3}\int_{\mathbb{R}^3}\partial_j(\nabla\cdot\mathbf{w})\partial_j(\nabla\cdot\mathbf{w})\,dx \\
&= -(\nabla(\nabla\cdot\mathbf{w}),\nabla(\nabla\cdot\mathbf{w}))_2 \\
&= -\|\nabla(\nabla\cdot\mathbf{w})\|_2^2.
\end{aligned}
$$

Além disso,

$$(\mathbf{w}, \Delta\mathbf{w})_2 = \sum_{i=1}^{3}(\mathbf{w}, \partial_i^2\mathbf{w})_2 = -\sum_{i=1}^{3}(\partial_i\mathbf{w}, \partial_i\mathbf{w})_2 = -(\nabla\mathbf{w}, \nabla\mathbf{w})_2 = -\|\nabla\mathbf{w}\|_2^2.$$

Dessa forma, chegamos a

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{w}\|_2^2 + \gamma\|\Delta\mathbf{w}\|_2^2 + \kappa\|\nabla(\nabla\cdot\mathbf{w})\|_2^2 + 2\chi\|\nabla\mathbf{w}\|_2^2 = \int_{\mathbb{R}^3}\mathbf{u}\cdot\nabla\mathbf{w}\cdot\Delta\mathbf{w}\,dx - \chi\int_{\mathbb{R}^3}(\nabla\times\mathbf{u})\cdot\Delta\mathbf{w}\,dx \tag{2.51}$$

Aplicando o produto $(\cdot, \Delta\mathbf{b})_2$ à terceira equação do sistema (1), conclui-se

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{b}\|_2^2 &= (\nabla\partial_t\mathbf{b}, \nabla\mathbf{b})_2 \\
&= -(\partial_t\mathbf{b}, \Delta\mathbf{b})_2 \\
&= -\nu\|\Delta\mathbf{b}\|_2^2 + (\mathbf{u}\cdot\nabla\mathbf{b}, \Delta\mathbf{b})_2 - (\mathbf{b}\cdot\nabla\mathbf{u}, \Delta\mathbf{b})_2,
\end{aligned}$$

ou equivalentemente,

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{b}\|_2^2 + \nu\|\Delta\mathbf{b}\|_2^2 = \int_{\mathbb{R}^3}(\mathbf{u}\cdot\nabla\mathbf{b})\cdot\Delta\mathbf{b}\,dx - \int_{\mathbb{R}^3}(\mathbf{b}\cdot\nabla\mathbf{u})\cdot\Delta\mathbf{b}\,dx. \tag{2.52}$$

Somando (2.50), (2.51) e (2.52), temos

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\nabla\mathbf{u}, \nabla\mathbf{w}, \nabla\mathbf{b})\|_2^2 + (\mu+\chi)\|\Delta\mathbf{u}\|_2^2 + \gamma\|\Delta\mathbf{w}\|_2^2 + \nu\|\Delta\mathbf{b}\|_2^2 + \kappa\|\nabla(\nabla\cdot\mathbf{w})\|_2^2 + 2\chi\|\nabla\mathbf{w}\|_2^2 \\
&= \int_{\mathbb{R}^3}(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\Delta\mathbf{u}\,dx - \int_{\mathbb{R}^3}(\mathbf{b}\cdot\nabla\mathbf{b})\cdot\Delta\mathbf{u}\,dx - \chi\int_{\mathbb{R}^3}(\nabla\times\mathbf{w})\cdot\Delta\mathbf{u}\,dx + \int_{\mathbb{R}^3}(\mathbf{u}\cdot\nabla\mathbf{w})\cdot\Delta\mathbf{w}\,dx \\
&- \chi\int_{\mathbb{R}^3}(\nabla\times\mathbf{u})\cdot\Delta\mathbf{w}\,dx + \int_{\mathbb{R}^3}(\mathbf{u}\cdot\nabla\mathbf{b})\cdot\Delta\mathbf{b}\,dx - \int_{\mathbb{R}^3}(\mathbf{b}\cdot\nabla\mathbf{u})\cdot\Delta\mathbf{b}\,dx.
\end{aligned} \tag{2.53}$$

Note que, por integração por partes, obtemos

$$
\begin{aligned}
&-\chi\int_{\mathbb{R}^3}(\nabla\times\mathbf{w})\cdot\Delta\mathbf{u}\,dx-\chi\int_{\mathbb{R}^3}(\nabla\times\mathbf{u})\cdot\Delta\mathbf{w}\,dx\\
&=-\chi\int_{\mathbb{R}^3}(\partial_1w_2\Delta u_3+\partial_2w_3\Delta u_1+\partial_3w_1\Delta u_2-\partial_3w_2\Delta u_1-\partial_1w_3\Delta u_2-\partial_2w_1\Delta u_3)\,dx\\
&\quad-\chi\int_{\mathbb{R}^3}(\partial_1u_2\Delta w_3+\partial_2u_3\Delta w_1+\partial_3u_1\Delta w_2-\partial_3u_2\Delta w_1-\partial_1u_3\Delta w_2-\partial_2u_1\Delta w_3)\,dx\\
&=\;-\chi\int_{\mathbb{R}^3}(\partial_1w_2\Delta u_3+\partial_2w_3\Delta u_1+\partial_3w_1\Delta u_2-\partial_3w_2\Delta u_1-\partial_1w_3\Delta u_2-\partial_2w_1\Delta u_3)\,dx\\
&\quad-\chi\int_{\mathbb{R}^3}(-\partial_1w_3\Delta u_2-\partial_2w_1\Delta u_3-\partial_3w_2\Delta u_1+\partial_3w_1\Delta u_2+\partial_1w_2\Delta u_3+\partial_2w_3\Delta u_1)\,dx\\
&=\;-2\chi\int_{\mathbb{R}^3}(\nabla\times\mathbf{w})\cdot\Delta\mathbf{u}\,dx.
\end{aligned}
$$

Assim, usando as Desigualdades de Cauchy e Young, obtemos

$$
-\chi\int_{\mathbb{R}^3}(\nabla\times\mathbf{w})\cdot\Delta\mathbf{u}\,dx-\chi\int_{\mathbb{R}^3}(\nabla\times\mathbf{u})\cdot\Delta\mathbf{w}\,dx\le 2\chi\|\nabla\times\mathbf{w}\|_2\,\|\Delta\mathbf{u}\|_2\le\chi\|\nabla\times\mathbf{w}\|_2^2+\chi\|\Delta\mathbf{u}\|_2^2.
$$

Agora, aplicando a Identidade de Parseval, chegamos a

$$
\begin{aligned}
\|\nabla\times\mathbf{w}\|_2^2&=\int_{\mathbb{R}^3}|\nabla\times\mathbf{w}|^2\,dk=\int_{\mathbb{R}^3}|\widehat{\nabla\times\mathbf{w}}|^2\,dk\\
&=\int_{\mathbb{R}^3}|ik\times\widehat{\mathbf{w}}|^2\,dk=\int_{\mathbb{R}^3}|k\times\widehat{\mathbf{w}}|^2\,dk\\
&\le\;\int_{\mathbb{R}^3}|k|^2|\widehat{\mathbf{w}}|^2\,dk=\int_{\mathbb{R}^3}|\widehat{\nabla\mathbf{w}}|^2\,dk\\
&=\int_{\mathbb{R}^3}|\nabla\mathbf{w}|^2\,dk=\|\nabla\mathbf{w}\|_2^2.
\end{aligned}
$$

Portanto,

$$
-\chi\int_{\mathbb{R}^3}(\nabla\times\mathbf{w})\cdot\Delta\mathbf{u}\,dx-\chi\int_{\mathbb{R}^3}(\nabla\times\mathbf{u})\cdot\Delta\mathbf{w}\,dx\le\chi\|\nabla\mathbf{w}\|_2^2+\chi\|\Delta\mathbf{u}\|_2^2. \tag{2.54}
$$

Substituindo (2.54) em (2.53), temos

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^2+\mu\|\Delta\mathbf{u}\|_2^2+\gamma\|\Delta\mathbf{w}\|_2^2+\nu\|\Delta\mathbf{b}\|_2^2+\kappa\|\nabla(\nabla\cdot\mathbf{w})\|_2^2+\chi\|\nabla\mathbf{w}\|_2^2\\
&\le\int_{\mathbb{R}^3}(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\Delta\mathbf{u}\,dx-\int_{\mathbb{R}^3}(\mathbf{b}\cdot\nabla\mathbf{b})\cdot\Delta\mathbf{u}\,dx+\int_{\mathbb{R}^3}(\mathbf{u}\cdot\nabla\mathbf{w})\cdot\Delta\mathbf{w}\,dx+\int_{\mathbb{R}^3}(\mathbf{u}\cdot\nabla\mathbf{b})\cdot\Delta\mathbf{b}\,dx\\
&-\int_{\mathbb{R}^3}(\mathbf{b}\cdot\nabla\mathbf{u})\cdot\Delta\mathbf{b}\,dx.
\end{aligned} \tag{2.55}
$$

Agora, nosso interesse é estimar as integrais do lado direito da desigualdade acima. Assim sendo,

$$\begin{aligned}\int_{\mathbb{R}^3} (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\Delta\mathbf{u}\,dx &= \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_i u_j)(\partial_k^2 u_j)\,dx\\ &= -\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}(\partial_k u_i)(\partial_i u_j)(\partial_k u_j)\,dx - \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_k\partial_i u_j)(\partial_k u_j)\,dx.\end{aligned}$$

Observe que

$$\begin{aligned}-\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_k\partial_i u_j)(\partial_k u_j)\,dx &= \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}(\partial_i u_i)(\partial_k^2 u_j)\,dx + \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_i\partial_k u_j)(\partial_k u_j)\,dx\\ &= \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_i\partial_k u_j)(\partial_k u_j)\,dx,\end{aligned}$$

pois $\nabla\cdot\mathbf{u} = 0$. Por conseguinte,

$$\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_k\partial_i u_j)(\partial_k u_j)\,dx = 0.$$

Logo,

$$\int_{\mathbb{R}^3} (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\Delta\mathbf{u}\,dx = -\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}(\partial_k u_i)(\partial_i u_j)(\partial_k u_j)\,dx \le C\int_{\mathbb{R}^3}|\nabla\mathbf{u}|^3\,dx.$$

Seguindo as mesmas ideias temos que

$$\begin{aligned}\int_{\mathbb{R}^3} (\mathbf{u}\cdot\nabla\mathbf{b})\cdot\Delta\mathbf{b}\,dx &= \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_i b_j)(\partial_k^2 b_j)\,dx\\ &= -\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}(\partial_k u_i)(\partial_i b_j)(\partial_k b_j)\,dx - \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_k\partial_i b_j)(\partial_k b_j)\,dx.\end{aligned}$$

A última integral acima é nula. Com efeito,

$$\begin{aligned}-\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_k\partial_i b_j)(\partial_k b_j)\,dx &= \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}(\partial_i u_i)(\partial_k^2 b_j)\,dx + \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_i\partial_k b_j)(\partial_k b_j)\,dx\\ &= \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_i\partial_k b_j)(\partial_k b_j)\,dx,\end{aligned}$$

pois $\nabla \cdot \mathbf{u} = 0$. Deste modo, infere-se

$$-\sum_{i,j,k=1}^{3} \int_{\mathbb{R}^3} u_i(\partial_k \partial_i b_j)(\partial_k b_j)\, dx = 0.$$

Assim, podemos concluir que

$$\int_{\mathbb{R}^3} (\mathbf{u} \cdot \nabla \mathbf{b}) \cdot \Delta \mathbf{b}\, dx = -\sum_{i,j,k=1}^{3} \int_{\mathbb{R}^3} (\partial_k u_i)(\partial_i b_j)(\partial_k b_j)\, dx \leq C \int_{\mathbb{R}^3} |\nabla \mathbf{u}||\nabla \mathbf{b}|^2\, dx.$$

Analogamente, obtem-se

$$\int_{\mathbb{R}^3} (\mathbf{u} \cdot \nabla \mathbf{w}) \cdot \Delta \mathbf{w}\, dx \leq C \int_{\mathbb{R}^3} |\nabla \mathbf{u}|\ |\nabla \mathbf{w}|^2\, dx.$$

Veja ainda que

$$\begin{aligned}
-\int_{\mathbb{R}^3} (\mathbf{b} \cdot \nabla \mathbf{b}) \cdot \Delta \mathbf{u}\, dx &= -\sum_{i,j,k=1}^{3} \int_{\mathbb{R}^3} b_i(\partial_i b_j)(\partial_k^2 u_j)\, dx \\
&= \sum_{i,j,k=1}^{3} \int_{\mathbb{R}^3} (\partial_k b_i)(\partial_i b_j)(\partial_k u_j)\, dx + \sum_{i,j,k=1}^{3} \int_{\mathbb{R}^3} b_i(\partial_k \partial_i b_j)(\partial_k u_j)\, dx.
\end{aligned}$$

Por integração por partes e usando o fato que $\nabla \cdot \mathbf{b} = 0$ é fácil encontrar o seguinte resultado para a última integral exposta acima.

$$\begin{aligned}
\sum_{i,j,k=1}^{3} \int_{\mathbb{R}^3} b_i(\partial_k \partial_i b_j)(\partial_k u_j)\, dx &= -\sum_{i,j,k=1}^{3} \int_{\mathbb{R}^3} (\partial_i b_i)(\partial_k u_j)(\partial_k b_j)\, dx - \sum_{i,j,k=1}^{3} \int_{\mathbb{R}^3} b_i(\partial_i \partial_k u_j)(\partial_k b_j)\, dx \\
&= -\sum_{i,j,k=1}^{3} \int_{\mathbb{R}^3} b_i(\partial_i \partial_k u_j)(\partial_k b_j)\, dx.
\end{aligned}$$

Consequentemente, chegamos a

$$-\int_{\mathbb{R}^3} (\mathbf{b} \cdot \nabla \mathbf{b}) \cdot \Delta \mathbf{u}\, dx \leq\ C \int_{\mathbb{R}^3} |\nabla \mathbf{b}|^2 |\nabla \mathbf{u}|\, dx - \sum_{i,j,k=1}^{3} \int_{\mathbb{R}^3} b_i(\partial_i \partial_k u_j)(\partial_k b_j)\, dx. \tag{2.56}$$

Também é verdade que

$$
\begin{aligned}
-\int_{\mathbb{R}^3} (\mathbf{b}\cdot\nabla\mathbf{u})\cdot\Delta\mathbf{b}\,dx &= -\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} b_i(\partial_i u_j)(\partial_k^2 b_j)\,dx \\
&= \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} (\partial_k b_i)(\partial_i u_j)(\partial_k b_j)\,dx + \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} b_i(\partial_k\partial_i u_j)(\partial_k b_j)\,dx \\
&\le C\int_{\mathbb{R}^3} |\nabla\mathbf{b}|^2|\nabla\mathbf{u}|\,dx + \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} b_i(\partial_k\partial_i u_j)(\partial_k b_j)\,dx. \qquad (2.57)
\end{aligned}
$$

Assim, somando (2.56) a (2.57), encontramos

$$
-\int_{\mathbb{R}^3} (\mathbf{b}\cdot\nabla\mathbf{b})\cdot\Delta\mathbf{u}\,dx - \int_{\mathbb{R}^3} (\mathbf{b}\cdot\nabla\mathbf{u})\cdot\Delta\mathbf{b}\,dx \le C\int_{\mathbb{R}^3} |\nabla\mathbf{b}|^2|\nabla\mathbf{u}|\,dx.
$$

Substituindo estes resultados acima na desigualdade (2.55), chegamos a

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^2 + \mu\|\Delta\mathbf{u}\|_2^2 + \gamma\|\Delta\mathbf{w}\|_2^2 + \nu\|\Delta\mathbf{b}\|_2^2 + \kappa\|\nabla(\nabla\cdot\mathbf{w})\|_2^2 + \chi\|\nabla\mathbf{w}\|_2^2 \\
&\le C\int_{\mathbb{R}^3} |\nabla\mathbf{u}|^3\,dx + C\int_{\mathbb{R}^3} |\nabla\mathbf{b}|^2\ |\nabla\mathbf{u}|\,dx + C\int_{\mathbb{R}^3} |\nabla\mathbf{u}|\ |\nabla\mathbf{w}|^2\,dx \\
&=: J_1 + J_2 + J_3. \qquad (2.58)
\end{aligned}
$$

No que segue, encotraremos estimativas para J_i, para $i = 1, 2, 3$. Note que, pelo Lema 2.2, obtemos

$$
\begin{aligned}
J_1 &:= C\int_{\mathbb{R}^3} |\nabla\mathbf{u}|^3\,dx \\
&\le C\|\nabla\mathbf{u}\|_2^{\frac{3}{2}}\ \|\nabla\partial_1\mathbf{u}\|_2^{\frac{1}{2}}\ \|\nabla\partial_2\mathbf{u}\|_2^{\frac{1}{2}}\ \|\nabla\partial_3\mathbf{u}\|_2^{\frac{1}{2}} \\
&\le C\|\nabla\nabla\mathbf{u}\|_2\ \|\nabla\mathbf{u}\|_2^{\frac{3}{2}}\ \|\nabla\partial_3\mathbf{u}\|_2^{\frac{1}{2}}.
\end{aligned}
$$

Usando a Desigualdade de Young, conclui-se

$$
\begin{aligned}
J_1 &\le \frac{\mu}{6}\|\nabla\nabla\mathbf{u}\|_2^2 + C\|\nabla\mathbf{u}\|_2^3\ \|\nabla\partial_3\mathbf{u}\|_2 \\
&= \frac{\mu}{6}\|\nabla\nabla\mathbf{u}\|_2^2 + C\|\nabla\mathbf{u}\|_2^2(\|\nabla\mathbf{u}\|_2\ \|\nabla\partial_3\mathbf{u}\|_2) \\
&\le \frac{\mu}{6}\|\nabla\nabla\mathbf{u}\|_2^2 + C\|\nabla\mathbf{u}\|_2^2(\|\nabla\mathbf{u}\|_2^2 + \|\nabla\partial_3\mathbf{u}\|_2^2). \qquad (2.59)
\end{aligned}
$$

Pela Desigualdade de Hölder, obtemos

$$
J_2 := C\int_{\mathbb{R}^3} |\nabla\mathbf{b}|^2\ |\nabla\mathbf{u}|\,dx \le C\|\nabla\mathbf{u}\|_3\|\nabla\mathbf{b}\|_3^2.
$$

Portanto, pelo Lema 2.2, temos que

$$\begin{aligned}
J_2 \leq\ & C\|\nabla \mathbf{u}\|_2^{\frac{1}{2}}\ \|\nabla\partial_1 \mathbf{u}\|_2^{\frac{1}{6}}\ \|\nabla\partial_2 \mathbf{u}\|_2^{\frac{1}{6}}\ \|\nabla\partial_3 \mathbf{u}\|_2^{\frac{1}{6}}\ \|\nabla \mathbf{b}\|_2\ \|\nabla\partial_1 \mathbf{b}\|_2^{\frac{1}{3}}\ \|\nabla\partial_2 \mathbf{b}\|_2^{\frac{1}{3}}\ \|\nabla\partial_3 \mathbf{b}\|_2^{\frac{1}{3}} \\
\leq\ & C\|\nabla \mathbf{u}\|_2^{\frac{1}{2}}\ \|\nabla\nabla \mathbf{u}\|_2^{\frac{1}{3}}\ \|\nabla\partial_3 \mathbf{u}\|_2^{\frac{1}{6}}\ \|\nabla \mathbf{b}\|_2\ \|\nabla\nabla \mathbf{b}\|_2^{\frac{2}{3}}\ \|\nabla\partial_3 \mathbf{b}\|_2^{\frac{1}{3}}.
\end{aligned}$$

Usando a Desigualdade de Young, obtemos

$$\begin{aligned}
J_2 &\leq \frac{\mu}{6}\|\nabla\nabla \mathbf{u}\|_2^2 + C\|\nabla \mathbf{u}\|_2^{\frac{3}{5}}\ \|\nabla\partial_3 \mathbf{u}\|_2^{\frac{1}{5}}\ \|\nabla \mathbf{b}\|_2^{\frac{6}{5}}\ \|\nabla\nabla \mathbf{b}\|_2^{\frac{4}{5}}\ \|\nabla\partial_3 \mathbf{b}\|_2^{\frac{2}{5}} \\
&\leq\ \frac{\mu}{6}\|\nabla\nabla \mathbf{u}\|_2^2 + \frac{\nu}{2}\|\nabla\nabla \mathbf{b}\|_2^2 + C\|\nabla \mathbf{u}\|_2\ \|\nabla\partial_3 \mathbf{u}\|_2^{\frac{1}{3}}\ \|\nabla \mathbf{b}\|_2^2\ \|\nabla\partial_3 \mathbf{b}\|_2^{\frac{2}{3}} \\
&\leq\ \frac{\mu}{6}\|\nabla\nabla \mathbf{u}\|_2^2 + \frac{\nu}{2}\|\nabla\nabla \mathbf{b}\|_2^2 + C\|\nabla \mathbf{b}\|_2^2(\|\nabla \mathbf{u}\|_2^2 + \|\nabla\partial_3 \mathbf{u}\|_2^2 + \|\nabla\partial_3 \mathbf{b}\|_2^2)
\end{aligned} \tag{2.60}$$

Analogamente ao que foi feito para estimar J_2, podemos encontrar as estimativas abaixo (basta aplicar as Desigualdades de Hölder e Young e o Lema 2.2):

$$\begin{aligned}
J_3 :&= C\int_{\mathbb{R}^3} |\nabla \mathbf{u}|\ |\nabla \mathbf{w}|^2\, dx \\
&= C\|\nabla \mathbf{u}\|_3\|\nabla \mathbf{w}\|_3^2 \\
&\leq \frac{\mu}{6}\|\nabla\nabla \mathbf{u}\|_2^2 + C\|\nabla \mathbf{u}\|_2^{\frac{3}{5}}\ \|\nabla\partial_3 \mathbf{u}\|_2^{\frac{1}{5}}\|\nabla \mathbf{w}\|_2^{\frac{6}{5}}\|\nabla\nabla \mathbf{w}\|_2^{\frac{4}{5}}\ \|\nabla\partial_3 \mathbf{w}\|_2^{\frac{2}{5}} \\
&\leq\ \frac{\mu}{6}\|\nabla\nabla \mathbf{u}\|_2^2 + \frac{\gamma}{2}\|\nabla\nabla \mathbf{w}\|_2^2 + C\|\nabla \mathbf{w}\|_2^2(\|\nabla \mathbf{u}\|_2\ \|\nabla\partial_3 \mathbf{u}\|_2^{\frac{1}{3}}\ \|\nabla\partial_3 \mathbf{w}\|_2^{\frac{2}{3}}) \\
&\leq\ \frac{\mu}{6}\|\nabla\nabla \mathbf{u}\|_2^2 + \frac{\gamma}{2}\|\nabla\nabla \mathbf{w}\|_2^2 + C\|\nabla \mathbf{w}\|_2^2(\|\nabla \mathbf{u}\|_2^2 + \|\nabla\partial_3 \mathbf{u}\|_2^2 + \|\nabla\partial_3 \mathbf{w}\|_2^2).
\end{aligned} \tag{2.61}$$

Substituindo (2.59), (2.60) e (2.61) em (2.58), obtemos

$$\begin{aligned}
&\frac{d}{dt}\|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})\|_2^2 + \mu\|\Delta \mathbf{u}\|_2^2 + \gamma\|\Delta \mathbf{w}\|_2^2 + \nu\|\Delta \mathbf{b}\|_2^2 + 2\kappa\|\nabla(\nabla\cdot \mathbf{w})\|_2^2 + 2\chi\|\nabla \mathbf{w}\|_2^2 \\
&\leq C\|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})\|_2^2(\|\nabla \mathbf{u}\|_2^2 + \|(\nabla\partial_3 \mathbf{u}, \nabla\partial_3 \mathbf{w}, \nabla\partial_3 \mathbf{b})\|_2^2).
\end{aligned}$$

Por fim, aplicando o Lema de Gronwall, (2.41), (2.49) e (1.1), chegamos a

$$\begin{aligned}
\|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})(\cdot,t)\|_2^2 &\leq \|(\nabla \mathbf{u}_0, \nabla \mathbf{w}_0, \nabla \mathbf{b}_0)\|_2^2 e^{C\int_0^t (\|\nabla \mathbf{u}(\cdot,s)\|_2^2 + \|(\nabla\partial_3 \mathbf{u}, \nabla\partial_3 \mathbf{w}, \nabla\partial_3 \mathbf{b})(\cdot,s)\|_2^2)\, ds} \\
&\leq C, \quad \forall\, t \in [0,T].
\end{aligned}$$

Pela teoria clássica para soluções do sistema (1), (no caso de Navier-Stokes, ver [33]), concluímos que $(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot,t)$ pode ser estendida suavemente além de $t = T$. $\square$

2.2 Critério de Regularidade Envolvendo Somente $\mathbf{u}(\cdot,t)$

Nesta seção, estamos interessados em demonstrar que uma solução fraca $(\mathbf{u},\mathbf{w},\mathbf{b})(\cdot,t)$ do sistema magneto-micropolar (1), definida em $[0,T]$, pode ser estendida suavemente além de T, quando a seguinte hipótese, envolvendo somente o campo velocidade $\mathbf{u}(\cdot,t)$, é assumida:

$$\mathbf{u} \in L^q(0,T;L^p(\mathbb{R}^3)), \quad \frac{2}{q}+\frac{3}{p} \leq 1,\ 3 < p \leq \infty.$$

Mais especificamente, estabeleceremos minuciosamente o seguinte teorema, o qual foi provado em 2010 por Y. Baoquan [1].

Teorema 2.2 (ver [1])**.** *Seja* $(\mathbf{u_0},\mathbf{w_0},\mathbf{b_0}) \in H^1(\mathbb{R}^3)$ *tal que* $\nabla\cdot\mathbf{u}_0 = \nabla\cdot\mathbf{b}_0 = 0$. *Assuma que* $(\mathbf{u},\mathbf{w},\mathbf{b})(\cdot,t) \in C[0,T;H^1(\mathbb{R}^3)) \cap C(0,T;H^2(\mathbb{R}^3))$ *é uma solução suave para o sistema* (1). *Se* $(\mathbf{u},\mathbf{w},\mathbf{b})(\cdot,t)$ *satisfaz*

$$\mathbf{u} \in L^q(0,T;L^p(\mathbb{R}^3)), \quad \frac{2}{q}+\frac{3}{p} \leq 1,\ 3 < p \leq \infty, \tag{2.62}$$

então a solução $(\mathbf{u},\mathbf{w},\mathbf{b})(\cdot,t)$ *pode ser suavemente estendida além de* $t=T$.

Demonstração. Para a demonstração desse teorema, iremos considerar dois casos para p.

Caso 1: Assuma que $3 < p < \infty$.

Estamos, primeiramente, interessados em provar uma estimativa para a norma-L^2 da i-ésima componente do gradiente da solução $(\mathbf{u},\mathbf{w},\mathbf{b})$. Assim sendo, considere esta derivada na primeira equação do sistema (1) em ordem a obter

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\partial_i\mathbf{u}\|_2^2 &= (\partial_i\partial_t\mathbf{u},\partial_i\mathbf{u})_2 \\
&= (\mu+\chi)(\Delta\partial_i\mathbf{u},\partial_i\mathbf{u})_2 - (\partial_i\mathbf{u}\cdot\nabla\mathbf{u},\partial_i\mathbf{u})_2 - (\mathbf{u}\cdot\nabla\partial_i\mathbf{u},\partial_i\mathbf{u})_2 + (\partial_i\mathbf{b}\cdot\nabla\mathbf{b},\partial_i\mathbf{u})_2 \\
&\quad + (\mathbf{b}\cdot\nabla\partial_i\mathbf{b},\partial_i\mathbf{u})_2 - (\nabla\partial_i(p+\frac{1}{2}|\mathbf{b}|^2)_2,\partial_i\mathbf{u})_2 + \chi(\nabla\times\partial_i\mathbf{w},\partial_i\mathbf{u})_2.
\end{aligned}$$

Permita-nos estudar algumas parcelas encontradas no lado diereito das igualdades acima. Dessa forma, note que

$$(\Delta\partial_i \mathbf{u}, \partial_i \mathbf{u})_2 = \sum_{j=1}^{3} ({\partial_j}^2 \partial_i \mathbf{u}, \partial_i \mathbf{u})_2 = -\sum_{j=1}^{3} (\partial_j \partial_i \mathbf{u}, \partial_j \partial_i \mathbf{u})_2$$
$$= -\sum_{j=1}^{3} \|\partial_j \partial_i \mathbf{u}\|_2^2. \tag{2.63}$$

Também temos que

$$\begin{aligned}
(\mathbf{u} \cdot \nabla\partial_i \mathbf{u}, \partial_i \mathbf{u})_2 &= \sum_{j,k=1}^{3} \int_{\mathbb{R}^3} u_j (\partial_j \partial_i u_k)(\partial_i u_k)\, dx \\
&= -\sum_{j,k=1}^{3} \int_{\mathbb{R}^3} (\partial_j u_j)(\partial_i u_k)(\partial_i u_k)\, dx - \sum_{j,k=1}^{3} \int_{\mathbb{R}^3} u_j (\partial_j \partial_i u_k)(\partial_i u_k)\, dx \\
&= -\sum_{j,k=1}^{3} \int_{\mathbb{R}^3} u_j (\partial_j \partial_i u_k)(\partial_i u_k)\, dx,
\end{aligned}$$

pois, $\nabla \cdot \mathbf{u} = 0$. Consequentemente,

$$(\mathbf{u} \cdot \nabla\partial_i \mathbf{u}, \partial_i \mathbf{u})_2 = 0. \tag{2.64}$$

Observe ainda que

$$\begin{aligned}
-(\nabla\partial_i (p + \frac{1}{2}|\mathbf{b}|^2), \partial_i \mathbf{u})_2 &= -\sum_{j=1}^{3} (\partial_j \partial_i (p + \frac{1}{2}|\mathbf{b}|^2), \partial_i u_j)_2 \\
&= \sum_{j=1}^{3} (\partial_i (p + \frac{1}{2}|\mathbf{b}|^2), \partial_j \partial_i u_j)_2 \\
&= 0,
\end{aligned}$$

já que $\mathbf{u}$ é livre de divergente. Deste modo,

$$\frac{1}{2}\frac{d}{dt}\|\partial_i \mathbf{u}\|_2^2 + (\mu + \chi)\sum_{j=1}^{3} \|\partial_j \partial_i \mathbf{u}\|_2^2 = -(\partial_i \mathbf{u} \cdot \nabla \mathbf{u}, \partial_i \mathbf{u})_2 + (\partial_i \mathbf{b} \cdot \nabla \mathbf{b}, \partial_i \mathbf{u})_2 + (\mathbf{b} \cdot \nabla\partial_i \mathbf{b}, \partial_i \mathbf{u})_2$$
$$+ \chi(\nabla \times \partial_i \mathbf{w}, \partial_i \mathbf{u})_2. \tag{2.65}$$

Derivando a segunda equação do sistema (1) com relação a i-ésima variável espacial, chegamos a

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_i \mathbf{w}\|_2^2 &= (\partial_t\partial_i \mathbf{w}, \partial_i \mathbf{w})_2 \\ &= \gamma(\Delta\partial_i \mathbf{w}, \partial_i \mathbf{w})_2 + \kappa(\nabla(\nabla\cdot\partial_i \mathbf{w}), \partial_i \mathbf{w})_2 - 2\chi(\partial_i \mathbf{w}, \partial_i \mathbf{w})_2 - (\partial_i \mathbf{u}\cdot\nabla \mathbf{w}, \partial_i \mathbf{w})_2 \\ &\quad - (\mathbf{u}\cdot\nabla\partial_i \mathbf{w}, \partial_i \mathbf{w})_2 + \chi(\nabla\times\partial_i \mathbf{u}, \partial_i \mathbf{w})_2.\end{aligned}$$

Note que, por integração por partes, obtemos

$$\begin{aligned}(\nabla(\nabla\cdot\partial_i \mathbf{w}), \partial_i \mathbf{w})_2 &= \sum_{j=1}^{3}(\partial_j(\nabla\cdot\partial_i \mathbf{w}), \partial_i w_j)_2 \\ &= -\sum_{j=1}^{3}(\nabla\cdot\partial_i \mathbf{w}, \partial_j\partial_i w_j)_2 \\ &= -(\nabla\cdot\partial_i \mathbf{w}, \nabla\cdot\partial_i \mathbf{w})_2 \\ &= -\|\nabla\cdot\partial_i \mathbf{w}\|_2^2.\end{aligned}$$

Analogamente ao que foi feito em (2.63) e (2.64), concluimos que

$$(\Delta\partial_i \mathbf{w}, \partial_i \mathbf{w})_2 = -\sum_{j=1}^{3}\|\partial_j\partial_i \mathbf{w}\|_2^2$$

e também

$$(\mathbf{u}\cdot\nabla\partial_i \mathbf{w}, \partial_i \mathbf{w})_2 = 0.$$

Desta forma,

$$\frac{1}{2}\frac{d}{dt}\|\partial_i \mathbf{w}\|_2^2 + \gamma\sum_{j=1}^{3}\|\partial_j\partial_i \mathbf{w}\|_2^2 + \kappa\|\nabla\cdot\partial_i \mathbf{w}\|_2^2 + 2\chi\|\partial_i \mathbf{w}\|_2^2 = -(\partial_i \mathbf{u}\cdot\nabla \mathbf{w}, \partial_i \mathbf{w})_2 + \chi(\nabla\times\partial_i \mathbf{u}, \partial_i \mathbf{w})_2. \tag{2.66}$$

E, por fim, aplicando o operador ∂_i à terceira equação do sistema (1), encontramos

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_i \mathbf{b}\|_2^2 &= (\partial_t\partial_i \mathbf{b}, \partial_i \mathbf{b})_2 \\ &= \nu(\Delta\partial_i \mathbf{b}, \partial_i \mathbf{b})_2 - (\partial_i \mathbf{u}\cdot\nabla \mathbf{b}, \partial_i \mathbf{b})_2 - (\mathbf{u}\cdot\nabla\partial_i \mathbf{b}, \partial_i \mathbf{b})_2 + (\partial_i \mathbf{b}\cdot\nabla \mathbf{u}, \partial_i \mathbf{b})_2 \\ &\quad + (\mathbf{b}\cdot\nabla\partial_i \mathbf{u}, \partial_i \mathbf{b})_2.\end{aligned}$$

Analogamente ao que foi feito em (2.63) e (2.64), obtemos

$$(\Delta\partial_i \mathbf{b}, \partial_i \mathbf{b})_2 = -\sum_{j=1}^{3} \|\partial_j\partial_i \mathbf{b}\|_2^2$$

e também

$$(\mathbf{u}\cdot\nabla\partial_i \mathbf{b}, \partial_i \mathbf{b})_2 = 0.$$

Portanto,

$$\frac{1}{2}\frac{d}{dt}\|\partial_i \mathbf{b}\|_2^2 + \nu\sum_{j=1}^{3}\|\partial_j\partial_i \mathbf{b}\|_2^2 = -(\partial_i \mathbf{u}\cdot\nabla \mathbf{b}, \partial_i \mathbf{b})_2 + (\partial_i \mathbf{b}\cdot\nabla \mathbf{u}, \partial_i \mathbf{b})_2 + (\mathbf{b}\cdot\nabla\partial_i \mathbf{u}, \partial_i \mathbf{b})_2. \tag{2.67}$$

Somando os resultados obtidos em (2.65), (2.66) e (2.67), infere-se

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\partial_i \mathbf{u}, \partial_i \mathbf{w}, \partial_i \mathbf{b})\|_2^2 + (\mu+\chi)\sum_{j=1}^{3}\|\partial_j\partial_i \mathbf{u}\|_2^2 + \gamma\sum_{j=1}^{3}\|\partial_j\partial_i \mathbf{w}\|_2^2 + \nu\sum_{j=1}^{3}\|\partial_j\partial_i \mathbf{b}\|_2^2 + \kappa\|\nabla\cdot\partial_i \mathbf{w}\|_2^2 \\
&+ 2\chi\|\partial_i \mathbf{w}\|_2^2 = -(\partial_i \mathbf{u}\cdot\nabla \mathbf{u}, \partial_i \mathbf{u})_2 + (\partial_i \mathbf{b}\cdot\nabla \mathbf{b}, \partial_i \mathbf{u})_2 + (\mathbf{b}\cdot\nabla\partial_i \mathbf{b}, \partial_i \mathbf{u})_2 + \chi(\nabla\times\partial_i \mathbf{w}, \partial_i \mathbf{u})_2 \\
&- (\partial_i \mathbf{u}\cdot\nabla \mathbf{w}, \partial_i \mathbf{w})_2 + \chi(\nabla\times\partial_i \mathbf{u}, \partial_i \mathbf{w})_2 - (\partial_i \mathbf{u}\cdot\nabla \mathbf{b}, \partial_i \mathbf{b})_2 + (\partial_i \mathbf{b}\cdot\nabla \mathbf{u}, \partial_i \mathbf{b})_2 + (\mathbf{b}\cdot\nabla\partial_i \mathbf{u}, \partial_i \mathbf{b})_2.
\end{aligned}$$

Agora, vamos estudar os termos do lado direito da igualdade acima. Primeiramente, observe que

$$\begin{aligned}
(\mathbf{b}\cdot\nabla\partial_i \mathbf{b}, \partial_i \mathbf{u})_2 + (\mathbf{b}\cdot\nabla\partial_i \mathbf{u}, \partial_i \mathbf{b})_2 &= \sum_{j,k=1}^{3}\int_{\mathbb{R}^3} b_j(\partial_j\partial_i b_k)(\partial_i u_k)\,dx + \sum_{j,k=1}^{3}\int_{\mathbb{R}^3} b_j(\partial_j\partial_i u_k)(\partial_i b_k)\,dx \\
&= -\sum_{j,k=1}^{3}\int_{\mathbb{R}^3} b_j(\partial_j\partial_i u_k)(\partial_i b_k)\,dx + \sum_{j,k=1}^{3}\int_{\mathbb{R}^3} b_j(\partial_j\partial_i u_k)(\partial_i b_k)\,dx \\
&= 0,
\end{aligned}$$

pois $\nabla\cdot\mathbf{b} = 0$. Analogamente ao que foi feito em (2.19), temos

$$\chi(\nabla\times\partial_i \mathbf{w}, \partial_i \mathbf{u})_2 + \chi(\nabla\times\partial_i \mathbf{u}, \partial_i \mathbf{w})_2 = 2\chi(\nabla\times\partial_i \mathbf{u}, \partial_i \mathbf{w})_2 \leq \chi\|\nabla\partial_i \mathbf{u}\|_2^2 + \chi\|\partial_i \mathbf{w}\|_2^2.$$

Assim, chegamos a

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\partial_i\mathbf{u},\partial_i\mathbf{w},\partial_i\mathbf{b})\|_2^2+\sum_{j=1}^{3}(\mu\|\partial_j\partial_i\mathbf{u}\|_2^2+\gamma\|\partial_j\partial_i\mathbf{w}\|_2^2+\nu\|\partial_j\partial_i\mathbf{b}\|_2^2)+\kappa\|\nabla\cdot\partial_i\mathbf{w}\|_2^2+\chi\|\partial_i\mathbf{w}\|_2^2\\
&\le|(\partial_i\mathbf{u}\cdot\nabla\mathbf{u},\partial_i\mathbf{u})_2|+|(\partial_i\mathbf{b}\cdot\nabla\mathbf{b},\partial_i\mathbf{u})_2|+|(\partial_i\mathbf{u}\cdot\nabla\mathbf{w},\partial_i\mathbf{w})_2|+|(\partial_i\mathbf{u}\cdot\nabla\mathbf{b},\partial_i\mathbf{b})_2|\\
&\quad+|(\partial_i\mathbf{b}\cdot\nabla\mathbf{u},\partial_i\mathbf{b})_2|\\
&=:L_1+L_2+L_3+L_4+L_5.
\end{aligned}
\tag{2.68}
$$

Agora, iremos estimar L_j, para $j=1,...,5$. Inicialmente vejamos que, por integração por partes e por utilizar o fato que $\nabla\cdot\mathbf{u}=0$, encontramos

$$
\begin{aligned}
L_1&:=|(\partial_i\mathbf{u}\cdot\nabla\mathbf{u},\partial_i\mathbf{u})_2|\\
&=\Big|\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}(\partial_i u_j)(\partial_j u_k)(\partial_i u_k)\,dx\Big|\\
&=\Big|\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}(\partial_j\partial_i u_j)(\partial_i u_k)u_k\,dx+\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}(\partial_i u_j)(\partial_j\partial_i u_k)u_k\,dx\Big|\\
&\le\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}|\partial_i u_j||\partial_j\partial_i u_k||u_k|\,dx\\
&\le\ C\int_{\mathbb{R}^3}|\mathbf{u}||\nabla\mathbf{u}||D^2\mathbf{u}|\,dx.
\end{aligned}
$$

Logo, pelas Desigualdades de Hölder e Gagliardo-Nirenberg, obtemos

$$
\begin{aligned}
L_1&\le C\|\mathbf{u}\|_p\|\nabla\mathbf{u}\|_{\frac{2p}{p-2}}\|D^2\mathbf{u}\|_2\\
&\le\ C\|\mathbf{u}\|_p\|D^2\mathbf{u}\|_2\|\nabla\mathbf{u}\|_2^{\frac{p-3}{p}}\|D^2\mathbf{u}\|_2^{\frac{3}{p}}\\
&=C\|\mathbf{u}\|_p\|\nabla\mathbf{u}\|_2^{\frac{p-3}{p}}\|D^2\mathbf{u}\|_2^{\frac{p+3}{p}}
\end{aligned}
$$

Analogamente,

$$\begin{aligned}
L_2 &:= |(\partial_i \mathbf{b} \cdot \nabla \mathbf{b}, \partial_i \mathbf{u})_2| \\
&= \Big| \sum_{j,k=1}^{3} \int_{\mathbb{R}^3} (\partial_i b_j)(\partial_j b_k)(\partial_i u_k)\, dx \Big| \\
&= \Big| \sum_{j,k=1}^{3} \int_{\mathbb{R}^3} (\partial_i \partial_i b_j)(\partial_j b_k) u_k \, dx + \sum_{j,k=1}^{3} \int_{\mathbb{R}^3} (\partial_i b_j)(\partial_i \partial_j b_k) u_k \, dx \Big| \\
&\leq C \int_{\mathbb{R}^3} |\mathbf{u}||\nabla \mathbf{b}||D^2 \mathbf{b}|\, dx \\
&\leq C \|\mathbf{u}\|_p \|\nabla \mathbf{b}\|_2^{\frac{p-3}{p}} \|D^2 \mathbf{b}\|_2^{\frac{p+3}{p}}
\end{aligned}$$

e também

$$\begin{aligned}
L_3 &:= |(\partial_i \mathbf{u} \cdot \nabla \mathbf{w}, \partial_i \mathbf{w})_2| \\
&= \Big| \sum_{j,k=1}^{3} \int_{\mathbb{R}^3} (\partial_i u_j)(\partial_j w_k)(\partial_i w_k)\, dx \Big| \\
&= \Big| \sum_{j,k=1}^{3} \int_{\mathbb{R}^3} (\partial_i \partial_j w_k)(\partial_i w_k) u_j \, dx + \sum_{j,k=1}^{3} \int_{\mathbb{R}^3} (\partial_i w_k)(\partial_i \partial_i w_k) u_j \, dx \Big| \\
&\leq C \int_{\mathbb{R}^3} |\mathbf{u}||\nabla \mathbf{w}||D^2 \mathbf{w}|\, dx \\
&\leq \; C \|\mathbf{u}\|_p \|\nabla \mathbf{w}\|_2^{\frac{p-3}{p}} \|D^2 \mathbf{w}\|_2^{\frac{p+3}{p}}.
\end{aligned}$$

Prosseguindo de forma semelhante, encontramos

$$L_4 \leq C \|\mathbf{u}\|_p \|\nabla \mathbf{b}\|_2^{\frac{p-3}{p}} \|D^2 \mathbf{b}\|_2^{\frac{p+3}{p}}$$

e também

$$L_5 \leq C \|\mathbf{u}\|_p \|\nabla \mathbf{b}\|_2^{\frac{p-3}{p}} \|D^2 \mathbf{b}\|_2^{\frac{p+3}{p}}.$$

Logo, passando a soma em (2.68), quando $i = 1, 2, 3$, obtemos

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})\|_2^2 + \mu \|D^2 \mathbf{u}\|_2^2 + \gamma \|D^2 \mathbf{w}\|_2^2 + \nu \|D^2 \mathbf{b}\|_2^2 + \kappa \|\nabla \cdot \nabla \mathbf{w}\|_2^2 + \chi \|\nabla \mathbf{w}\|_2^2 \\
&\leq C \|\mathbf{u}\|_p [\|\nabla \mathbf{u}\|_2^{\frac{p-3}{p}} \|D^2 \mathbf{u}\|_2^{\frac{p+3}{p}} + \|\nabla \mathbf{b}\|_2^{\frac{p-3}{p}} \|D^2 \mathbf{b}\|_2^{\frac{p+3}{p}} + \|\nabla \mathbf{w}\|_2^{\frac{p-3}{p}} \|D^2 \mathbf{w}\|_2^{\frac{p+3}{p}}].
\end{aligned}$$

Mas, pela Desigualdade de Young, temos que

$$C\|\mathbf{u}\|_p\|\nabla\mathbf{b}\|_2^{\frac{p-3}{p}}\|D^2\mathbf{b}\|_2^{\frac{p+3}{p}} \leq \frac{\nu}{2}\|D^2\mathbf{b}\|_2^2 + C\|\mathbf{u}\|_p^{\frac{2p}{p-3}}\|\nabla\mathbf{b}\|_2^2.$$

Analogamente, obtem-se

$$C\|\mathbf{u}\|_p\|\nabla\mathbf{u}\|_2^{\frac{p-3}{p}}\|D^2\mathbf{u}\|_2^{\frac{p+3}{p}} \leq \frac{\mu}{2}\|D^2\mathbf{u}\|_2^2 + C\|\mathbf{u}\|_p^{\frac{2p}{p-3}}\|\nabla\mathbf{u}\|_2^2$$

e também

$$C\|\mathbf{u}\|_p\|\nabla\mathbf{w}\|_2^{\frac{p-3}{p}}\|D^2\mathbf{w}\|_2^{\frac{p+3}{p}} \leq \frac{\gamma}{2}\|D^2\mathbf{w}\|_2^2 + C\|\mathbf{u}\|_p^{\frac{2p}{p-3}}\|\nabla\mathbf{w}\|_2^2.$$

Por conseguinte, infere-se

$$\begin{aligned}&\frac{d}{dt}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^2 + \mu\|D^2\mathbf{u}\|_2^2 + \gamma\|D^2\mathbf{w}\|_2^2 + \nu\|D^2\mathbf{b}\|_2^2 + 2\kappa\|\nabla\cdot\nabla\mathbf{w}\|_2^2 + 2\chi\|\nabla\mathbf{w}\|_2^2\\ &\leq C\|\mathbf{u}\|_p^{\frac{2p}{p-3}}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^2.\end{aligned}$$

Pelo Lema de Gronwall, chegamos a

$$\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\cdot,t)\|_2^2 \leq \|(\nabla\mathbf{u}_0,\nabla\mathbf{w}_0,\nabla\mathbf{b}_0)\|_2^2 e^{C\int_0^T \|\mathbf{u}(\cdot,\tau)\|_p^{\frac{2p}{p-3}}\,d\tau}, \quad \forall\, t\in[0,T].$$

Como

$$\frac{2}{\frac{2p}{p-3}} + \frac{3}{p} = \frac{2(p-3)}{2p} + \frac{3}{p} = \frac{2(p-3)+6}{2p} = 1,$$

então, pela hipótese (2.62), conclui-se

$$\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\cdot,t)\|_2 \leq C, \quad \forall\, t\in[0,T].$$

Por fim, pela teoria clássica envolvendo a solução $(\mathbf{u},\mathbf{w},\mathbf{b})(\cdot,t)$ do sistema magneto-micropolar (1), (no caso de Navier-Stokes, ver [33]), o resultado segue.

Caso 2: Considere agora que $p=\infty$.

Vamos estimar novamente L_j, $j=1,...,5$. Primeiramente, pela Desigualdade de Hölder, temos

$$L_1 \leq C\int_{\mathbb{R}^3}|\mathbf{u}||\nabla\mathbf{u}||D^2\mathbf{u}|\,dx \leq C\|\mathbf{u}\|_\infty\int_{\mathbb{R}^3}|\nabla\mathbf{u}||D^2\mathbf{u}|\,dx \leq C\|\mathbf{u}\|_\infty\|\nabla\mathbf{u}\|_2\|D^2\mathbf{u}\|_2.$$

Analogamente, é simples mostrar que

$$L_2 \leq C\|\mathbf{u}\|_\infty \|\nabla \mathbf{b}\|_2 \|D^2\mathbf{b}\|_2,$$

$$L_3 \leq C\|\mathbf{u}\|_\infty \|\nabla \mathbf{w}\|_2 \|D^2\mathbf{w}\|_2,$$

$$L_4 \leq C\|\mathbf{u}\|_\infty \|\nabla \mathbf{b}\|_2 \|D^2\mathbf{b}\|_2,$$

e também

$$L_5 \leq C\|\mathbf{u}\|_\infty \|\nabla \mathbf{b}\|_2 \|D^2\mathbf{b}\|_2.$$

Consequentemente, podemos escrever

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})\|_2^2 + \mu\|D^2\mathbf{u}\|_2^2 + \gamma\|D^2\mathbf{w}\|_2^2 + \nu\|D^2\mathbf{b}\|_2^2 + \kappa\|\nabla \cdot \nabla \mathbf{w}\|_2^2 + \chi\|\nabla \mathbf{w}\|_2^2\\ &\leq C\|\mathbf{u}\|_\infty[\|\nabla \mathbf{u}\|_2\|D^2\mathbf{u}\|_2 + \|\nabla \mathbf{b}\|_2\|D^2\mathbf{b}\|_2 + \|\nabla \mathbf{w}\|_2\ \|D^2\mathbf{w}\|_2].\end{aligned}$$

Logo, pela Desigualdade de Young, obtemos

$$\begin{aligned}&\frac{d}{dt}\|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})\|_2^2 + \mu\|D^2\mathbf{u}\|_2^2 + \gamma\|D^2\mathbf{w}\|_2^2 + \nu\|D^2\mathbf{b}\|_2^2 + 2\kappa\|\nabla \cdot \nabla \mathbf{w}\|_2^2 + 2\chi\|\nabla \mathbf{w}\|_2^2\\ &\leq C\|\mathbf{u}\|_\infty^2\|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})\|_2^2.\end{aligned}$$

Assim, pelo Lema de Gronwall, chegamos a

$$\|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})(\cdot, t)\|_2^2 \leq \|(\nabla \mathbf{u}_0, \nabla \mathbf{w}_0, \nabla \mathbf{b}_0)\|_2^2 e^{C\int_0^T \|\mathbf{u}(\cdot,\tau)\|_\infty^2\, d\tau}, \quad \forall\, t \in [0, T].$$

Então, pela hipótese (2.62), conclui-se que

$$\|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})(\cdot, t)\|_2 \leq C, \quad \forall\, t \in [0, T].$$

Por fim, pela teoria clássica envolvendo a solução $(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t)$ do sistema magneto-micropolar (1), (no caso de Navier-Stokes, ver [33]), o resultado segue. $\square$

2.3 Critério de Regularidade Envolvendo Somente $\nabla\mathbf{u}(\cdot,t)$

Nesta seção, nosso desejo é mostrar que uma solução fraca $(\mathbf{u},\mathbf{w},\mathbf{b})(\cdot,t)$ do sistema magneto-micropolar (1), definida em $[0,T]$, pode ser estendida suavemente além de T, quando a seguinte hipótese, envolvendo somente o gradiente do campo velocidade $\nabla\mathbf{u}(\cdot,t)$, é considerada:

$$\int_0^T \|\nabla\mathbf{u}(\cdot,t)\|_p^q\,dt < \infty, \quad \frac{3}{2} < p \leq \infty,\ \frac{2}{q}+\frac{3}{p} \leq 2.$$

Mais precisamente, provaremos o seguinte resultado, o qual foi estabelecido por Y. Baoquan [1] em 2010, sobre regularidade de soluções fracas para o sistema (1).

Teorema 2.3 (ver [1])**.** *Seja* $(\mathbf{u_0},\mathbf{w_0},\mathbf{b_0}) \in H^1(\mathbb{R}^3)$ *tal que* $\nabla\cdot\mathbf{u}_0 = \nabla\cdot\mathbf{b}_0 = 0$*. Assuma que* $(\mathbf{u},\mathbf{w},\mathbf{b})(\cdot,t) \in C[0,T;H^1(\mathbb{R}^3)) \cap C(0,T;H^2(\mathbb{R}^3))$ *é uma solução suave para o sistema* (1)*. Se* $(\mathbf{u},\mathbf{w},\mathbf{b})(\cdot,t)$ *satisfaz*

$$\int_0^T \|\nabla\mathbf{u}(\cdot,t)\|_p^q\,dt < \infty, \quad \frac{3}{2} < p \leq \infty,\ \frac{2}{q}+\frac{3}{p} \leq 2. \tag{2.69}$$

então a solução $(\mathbf{u},\mathbf{w},\mathbf{b})(\cdot,t)$ *pode ser suavemente estendida além de* $t=T$*.*

Demonstração. Separaremos a prova deste resultado em dois casos.

Caso 1: Considere que $\frac{3}{2} < p < \infty$.

Já vimos, no Teorema 2.2, desigualdade (2.68), que

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\partial_i\mathbf{u},\partial_i\mathbf{w},\partial_i\mathbf{b})\|_2^2 + \sum_{j=1}^{3}(\mu\|\partial_j\partial_i\mathbf{u}\|_2^2 + \gamma\|\partial_j\partial_i\mathbf{w}\|_2^2 + \nu\|\partial_j\partial_i\mathbf{b}\|_2^2) + \kappa\|\nabla\cdot\partial_i\mathbf{w}\|_2^2 + \chi\|\partial_i\mathbf{w}\|_2^2\\
&\leq |(\partial_i\mathbf{u}\cdot\nabla\mathbf{u},\partial_i\mathbf{u})_2| + |(\partial_i\mathbf{b}\cdot\nabla\mathbf{b},\partial_i\mathbf{u})_2| + |(\partial_i\mathbf{u}\cdot\nabla\mathbf{w},\partial_i\mathbf{w})_2| + |(\partial_i\mathbf{u}\cdot\nabla\mathbf{b},\partial_i\mathbf{b})_2|\\
&+ |(\partial_i\mathbf{b}\cdot\nabla\mathbf{u},\partial_i\mathbf{b})_2|\\
&=: Q_1+Q_2+Q_3+Q_4+Q_5.
\end{aligned}$$

Agora, iremos estimar Q_j, para $j=1,...,5$, de uma outra forma. Comecemos com Q_1.

$$Q_1 := |(\partial_i\mathbf{u}\cdot\nabla\mathbf{u},\partial_i\mathbf{u})_2| = \Big|\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}(\partial_i u_j)(\partial_j u_k)(\partial_i u_k)\,dx\Big| \leq C\int_{\mathbb{R}^3}|\nabla\mathbf{u}|^3\,dx.$$

Logo, pelas Desigualdades de Hölder e Gagliardo-Nirenberg, chegamos a

$$Q_1 \leq C\|\nabla \mathbf{u}\|_p \; \|\nabla \mathbf{u}\|^2_{\frac{2p}{p-1}} \leq C\|\nabla \mathbf{u}\|_p \; \|\nabla \mathbf{u}\|_2^{2-\frac{3}{p}} \|D^2\mathbf{u}\|_2^{\frac{3}{p}}.$$

Analogamente, é fácil ver que

$$Q_2 := |(\partial_i \mathbf{b} \cdot \nabla \mathbf{b}, \partial_i \mathbf{u})_2| \leq C \int_{\mathbb{R}^3} |\nabla \mathbf{u}| |\nabla \mathbf{b}|^2 \, dx \leq C\|\nabla \mathbf{u}\|_p \; \|\nabla \mathbf{b}\|_2^{2-\frac{3}{p}} \|D^2\mathbf{b}\|_2^{\frac{3}{p}},$$

$$Q_3 := |(\partial_i \mathbf{u} \cdot \nabla \mathbf{w}, \partial_i \mathbf{w})_2| \leq C \int_{\mathbb{R}^3} |\nabla \mathbf{u}| |\nabla \mathbf{w}|^2 \, dx \leq C\|\nabla \mathbf{u}\|_p \; \|\nabla \mathbf{w}\|_2^{2-\frac{3}{p}} \|D^2\mathbf{w}\|_2^{\frac{3}{p}},$$

$$Q_4 := |(\partial_i \mathbf{u} \cdot \nabla \mathbf{b}, \partial_i \mathbf{b})_2| \leq C \int_{\mathbb{R}^3} |\nabla \mathbf{u}| |\nabla \mathbf{b}|^2 \, dx \leq \; C\|\nabla \mathbf{u}\|_p \; \|\nabla \mathbf{b}\|_2^{2-\frac{3}{p}} \|D^2\mathbf{b}\|_2^{\frac{3}{p}}$$

e também

$$Q_5 \leq C\|\nabla \mathbf{u}\|_p \|\nabla \mathbf{b}\|_2^{2-\frac{3}{p}} \|D^2\mathbf{b}\|_2^{\frac{3}{p}}.$$

Logo, passando a soma em (2.68), quando $i = 1, 2, 3$, obtemos

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})\|_2^2 + \mu\|D^2\mathbf{u}\|_2^2 + \gamma\|D^2\mathbf{w}\|_2^2 + \nu\|D^2\mathbf{b}\|_2^2 + \kappa\|\nabla \cdot \nabla \mathbf{w}\|_2^2 + \chi\|\nabla \mathbf{w}\|_2^2 \\
&\leq C\|\nabla \mathbf{u}\|_p [\|\nabla \mathbf{u}\|_2^{\frac{2p-3}{p}} \|D^2\mathbf{u}\|_2^{\frac{3}{p}} + \|\nabla \mathbf{w}\|_2^{\frac{2p-3}{p}} \|D^2\mathbf{w}\|_2^{\frac{3}{p}} + \|\nabla \mathbf{b}\|_2^{\frac{2p-3}{p}} \|D^2\mathbf{b}\|_2^{\frac{3}{p}}].
\end{aligned}$$

Mas, pela Desigualdade de Young, temos que

$$C\|\nabla \mathbf{u}\|_p \|\nabla \mathbf{u}\|_2^{\frac{2p-3}{p}} \|D^2\mathbf{u}\|_2^{\frac{3}{p}} \leq \frac{\mu}{4}\|D^2\mathbf{u}\|_2^2 + C\|\nabla \mathbf{u}\|_p^{\frac{2p}{2p-3}} \|\nabla \mathbf{u}\|_2^2.$$

Analogamente, obtém-se

$$C\|\nabla \mathbf{u}\|_p \|\nabla \mathbf{w}\|_2^{\frac{2p-3}{p}} \|D^2\mathbf{w}\|_2^{\frac{3}{p}} \leq \frac{\gamma}{2}\|D^2\mathbf{w}\|_2^2 + C\|\nabla \mathbf{u}\|_p^{\frac{2p}{2p-3}} \|\nabla \mathbf{w}\|_2^2$$

e também

$$C\|\nabla \mathbf{u}\|_p \|\nabla \mathbf{b}\|_2^{\frac{2p-3}{p}} \|D^2\mathbf{b}\|_2^{\frac{3}{p}} \leq \frac{\nu}{2}\|D^2\mathbf{b}\|_2^2 + C\|\nabla \mathbf{u}\|_p^{\frac{2p}{2p-3}} \|\nabla \mathbf{b}\|_2^2.$$

Portanto, podemos escrever o seguinte:

$$
\begin{aligned}
&\frac{d}{dt}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^2+\mu\|D^2\mathbf{u}\|_2^2+\gamma\|D^2\mathbf{w}\|_2^2+\nu\|D^2\mathbf{b}\|_2^2+2\kappa\|\nabla\cdot\nabla\mathbf{w}\|_2^2+2\chi\|\nabla\mathbf{w}\|_2^2\\
&\leq C\|\nabla\mathbf{u}\|_p^{\frac{2p}{2p-3}}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^2.
\end{aligned}
$$

Pelo Lema de Gronwall, conclui-se

$$
\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\cdot,t)\|_2^2\leq\|(\nabla\mathbf{u}_0,\nabla\mathbf{w}_0,\nabla\mathbf{b}_0)\|_2^2 e^{C\int_0^T\|\nabla\mathbf{u}(\cdot,\tau)\|_p^{\frac{2p}{2p-3}}\,d\tau},\quad\forall\, t\in[0,T].
$$

Mas,

$$
\frac{2}{\frac{2p}{p-3}}+\frac{3}{p}=\frac{2p-6}{2p}+\frac{3}{p}=\frac{2p-6+6}{2p}=1\leq 2.
$$

Logo, por utilizar a hipótese (2.69), chegamos a

$$
\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\cdot,t)\|_2\leq C,\quad\forall\, t\in[0,T].
$$

Por fim, pela teoria clássica envolvendo a solução $(\mathbf{u},\mathbf{w},\mathbf{b})(\cdot,t)$ do sistema magneto-micropolar (1), (no caso de Navier-Stokes, ver [33]), o resultado segue.

Caso 2: Assuma que $p=\infty$.

Vamos estimar novamente Q_j $(j=1,...,5)$ utilizando uma abordagem levemente diferente. Assim sendo,

$$
Q_1\leq C\int_{\mathbb{R}^3}|\nabla\mathbf{u}||\nabla\mathbf{u}|^2\,dx\leq C\|\nabla\mathbf{u}\|_\infty\int_{\mathbb{R}^3}|\nabla\mathbf{u}|^2\,dx\leq C\|\nabla\mathbf{u}\|_\infty\|\nabla\mathbf{u}\|_2^2.
$$

Da mesma forma, temos

$$
Q_2\leq C\int_{\mathbb{R}^3}|\nabla\mathbf{u}||\nabla\mathbf{b}|^2\,dx\leq C\|\nabla\mathbf{u}\|_\infty\int_{\mathbb{R}^3}|\nabla\mathbf{b}|^2\,dx\leq C\|\nabla\mathbf{u}\|_\infty\|\nabla\mathbf{b}\|_2^2.
$$

Analogamente, podemos chegar a

$$
Q_3\leq C\|\nabla\mathbf{u}\|_\infty\|\nabla\mathbf{w}\|_2^2,
$$

$$
Q_4\leq C\|\nabla\mathbf{u}\|_\infty\|\nabla\mathbf{b}\|_2^2
$$

e também

$$Q_5 \leq C\|\nabla \mathbf{u}\|_\infty \|\nabla \mathbf{b}\|_2^2.$$

Portanto por (2.68), concluímos que

$$\begin{aligned}&\frac{d}{dt}\|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})\|_2^2 + 2\mu\|D^2\mathbf{u}\|_2^2 + 2\gamma\|D^2\mathbf{w}\|_2^2 + 2\nu\|D^2\mathbf{b}\|_2^2 + 2\kappa\|\nabla\cdot\nabla\mathbf{w}\|_2^2 + 2\chi\|\nabla\mathbf{w}\|_2^2 \\ &\leq C\|\nabla \mathbf{u}\|_\infty \|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})\|_2^2.\end{aligned}$$

Logo, pelo Lema de Gronwall, obtemos

$$\|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})(t)\|_2^2 \leq \|(\nabla \mathbf{u}_0, \nabla \mathbf{w}_0, \nabla \mathbf{b}_0)\|_2^2 e^{C\int_0^T \|\nabla \mathbf{u}(\cdot,\tau)\|_\infty \, d\tau}, \quad \forall\, t \in [0, T],$$

Por fim, pela teoria clássica envolvendo a solução $(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t)$ do sistema magneto-micropolar (1) o resultado segue. Aqui usamos o fato que $\displaystyle\int_0^T \|\nabla \mathbf{u}\|_\infty \, dt < \infty$ (ver (2.69)).

□

Em ordem a concluir este capítulo, é importante destacar que se $(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t)$ é uma solução fraca para o sistema magneto-micropolar (1) em $[0, \infty)$ (sobre existência deste tipo de solução citamos [47, 54] e referências inclusas) e $(\mathbf{u}', \mathbf{w}', \mathbf{b}')(\cdot, t)$ é a solução forte para o mesmo sistema no intervalo maximal $[0, T^*)$ satisfazendo $(\mathbf{u}', \mathbf{w}', \mathbf{b}')(\cdot, 0) = (\mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0) \in H^1(\mathbb{R}^3)$ (sobre existência deste tipo de solução listamos [46, 53] e referências inclusas), então $(\mathbf{u}', \mathbf{w}', \mathbf{b}')(\cdot, t) = (\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t)$ para todo $t \in [0, T^*)$; desde que, $(\mathbf{u}', \mathbf{w}', \mathbf{b}')(\cdot, t)$ é a única solução fraca de (1) em $[0, T^*)$. Portanto, se supusermos, por absurdo, que $T^* < T$ nos Teoremas 2.1, 2.2 e 2.3, assim as hipóteses (2.3), (2.62) e (2.69) são satisfeitas para $(\mathbf{u}', \mathbf{w}', \mathbf{b}')(\cdot, t)(= (\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t))$ em $[0, T^*)$. Consequentemente, estes teoremas garantem que $(\mathbf{u}', \mathbf{w}', \mathbf{b}')(\cdot, t)$ pode ser estendida suavemente além de $t = T^*$. Isto é uma contradição pela maximalidade deste tempo. Por fim, $T \leq T^*$. Deste modo, $(\mathbf{u}, \mathbf{w}, \mathbf{b})$ é suave em $\mathbb{R}^3 \times (0, T)$; já que, o mesmo ocorre com $(\mathbf{u}', \mathbf{w}', \mathbf{b}')$ em $\mathbb{R}^3 \times (0, T^*)$.

Capítulo 3

Critério de Regularidade Envolvendo uma Entrada do Campo Velocidade

Neste capítulo, mostraremos como é possível estender três resultados de regularidade para uma solução fraca das equações de Navier-Stokes (3) para as equações magneto-micropolares (1).

3.1 Critério de Regularidade Envolvendo $(\nabla u_3, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})(\cdot, t)$

Nesta seção, temos como objetivo mostrar mais um critério de regularidade para uma solução fraca $(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t)$ do sistema de equações magneto-micropolares (1). Com base em um resultado de regularidade para uma solução fraca $\mathbf{u}(\cdot, t)$ das equações de Navier-Stokes (3), o qual garante suavidade para $\mathbf{u}(\cdot, t)$ quando lhe é imposto a seguinte hipótese sobre a terceira coordenada do campo velocidade,

$$\nabla u_3(\cdot, t) \in L^{\frac{32}{7}}(0, T; L^2(\mathbb{R}^3)),$$

ver [73], conseguimos obter um resultado semelhante para $(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t)$, quando adotamos uma hipótese como segue:

$$(\nabla u_3, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})(\cdot, t) \in L^{\frac{32}{7}}(0, T; L^2(\mathbb{R}^3)).$$

Para este fim, exibimos um lema que desempenha um papel importante na prova da afirmação acima. Tal resultado foi demonstrado em [63].

Lema 3.1 (ver [63]). *Sejam i, j, k permutação de $1, 2, 3$. Assuma que $f, g, \partial_i g, \partial_j g, h, \partial_j h, \partial_k h \in L^2(\mathbb{R}^3)$. Então,*

$$\int_{\mathbb{R}^3} |f(x)g(x)h(x)|\, dx \leq C\|f\|_2 \|g\|_2^{\frac{1}{4}} \|\partial_i g\|_2^{\frac{1}{2}} \|\partial_j g\|_2^{\frac{1}{4}} \|h\|_2^{\frac{1}{4}} \|\partial_k h\|_2^{\frac{1}{2}} \|\partial_j h\|_2^{\frac{1}{4}},$$

onde C é uma constante positiva.

Demonstração. Sem perda de generalidade consideraremos que $f, g, h \in C_c^\infty(\mathbb{R}^3)$ e $i = 1, k = 2, j = 3$. Observe que, pela Desigualdade de Hölder, obtemos

$$\begin{aligned}
\int_{\mathbb{R}^3} |fgh|(x)\, dx &\leq \int_{\mathbb{R}^2} \sup_{x_1 \in \mathbb{R}} |g(x_1, x_2, x_3)| \left(\int_{\mathbb{R}} |fh|\, dx_1 \right) dx_2 dx_3 \\
&\leq \int_{\mathbb{R}^2} \sup_{x_1 \in \mathbb{R}} |g(x_1, x_2, x_3)| \left(\int_{\mathbb{R}} |f|^2\, dx_1 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |h|^2\, dx_1 \right)^{\frac{1}{2}} dx_2 dx_3 \\
&= \int_{\mathbb{R}^2} \sup_{x_1 \in \mathbb{R}} |g(x_1, x_2, x_3)| \|f(\cdot, x_2, x_3)\|_2 \|h(\cdot, x_2, x_3)\|_2\, dx_2 dx_3. \qquad (3.1)
\end{aligned}$$

Note que,

$$\partial_1(g^2)(x) = 2g(x)\partial_1 g(x).$$

Desta forma, integrando em $(-\infty, x_1)$, pelo Teorema Fundamental do Cálculo e Desigualdade de Hölder, chegamos a

$$\begin{aligned}
g^2(x) &= \int_{-\infty}^{x_1} \partial_1 g^2(s, x_2, x_3)\, ds \\
&= 2\int_{-\infty}^{x_1} g(s, x_2, x_3)\partial_1 g(s, x_2, x_3)\, ds \\
&\leq 2\int_{-\infty}^{x_1} |g(s, x_2, x_3)|\, |\partial_1 g(s, x_2, x_3)|\, ds \\
&\leq 2 \left(\int_{\mathbb{R}} |g(x_1, x_2, x_3)|^2\, dx_1 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |\partial_1 g(x_1, x_2, x_3)|^2\, dx_1 \right)^{\frac{1}{2}}.
\end{aligned}$$

Assim, podemos escrever

$$\sup_{x_1 \in \mathbb{R}} |g(x_1, x_2, x_3)| \leq \sqrt{2}\|g(\cdot, x_2, x_3)\|_2^{\frac{1}{2}} \|\partial_1 g(\cdot, x_2, x_3)\|_2^{\frac{1}{2}}.$$

Substituindo a desigualdade acima em (3.1), encontramos, através da Desigualdade de Hölder, que

$$\begin{aligned}\int_{\mathbb{R}^3} |fgh|(x)\,dx &\leq \sqrt{2}\int_{\mathbb{R}^2} \|g(\cdot,x_2,x_3)\|_2^{\frac{1}{2}}\,\|\partial_1 g(\cdot,x_2,x_3)\|_2^{\frac{1}{2}}\,\|f(\cdot,x_2,x_3)\|_2\|h(\cdot,x_2,x_3)\|_2\,dx_2dx_3\\ &\leq \sqrt{2}\int_{\mathbb{R}} \sup_{x_2\in\mathbb{R}} \|h(\cdot,x_2,x_3)\|_2\|g(\cdot,\cdot,x_3)\|_2^{\frac{1}{2}}\,\|\partial_1 g(\cdot,\cdot,x_3)\|_2^{\frac{1}{2}}\,\|f(\cdot,\cdot,x_3)\|_2\,dx_3. \qquad (3.2)\end{aligned}$$

Note que,

$$|h(x)|^2 = 2\int_{-\infty}^{x_2} h(x_1,s,x_3)\partial_2 h(x_1,s,x_3)\,ds \leq 2\int_{\mathbb{R}} |h(x_1,x_2,x_3)|\,|\partial_2 h(x_1,x_2,x_3)|\;dx_2.$$

Dessa forma, podemos inferir

$$\|h(\cdot,x_2,x_3)\|_2^2 \leq 2\int_{\mathbb{R}^2} |h(x_1,x_2,x_3)|\,|\partial_2 h(x_1,x_2,x_3)|\;dx_1dx_2.$$

Com isso, pela Desigualdade de Hölder, obtemos

$$\begin{aligned}\sup_{x_2\in\mathbb{R}} \|h(\cdot,x_2,x_3)\|_2 &\leq \sqrt{2}\left(\int_{\mathbb{R}} \|h(\cdot,x_2,x_3)\|_2\,\|\partial_2 h(\cdot,x_2,x_3)\|_2\;dx_2\right)^{\frac{1}{2}}\\ &\leq \sqrt{2}\|h(\cdot,\cdot,x_3)\|_2^{\frac{1}{2}}\,\|\partial_2 h(\cdot,\cdot,x_3)\|_2^{\frac{1}{2}}\,.\end{aligned}$$

Substituindo a desigualdade acima em (3.2), concluímos que

$$\begin{aligned}\int_{\mathbb{R}^3} |fgh|\,dx &\leq 2\int_{\mathbb{R}} \|h(\cdot,\cdot,x_3)\|_2^{\frac{1}{2}}\,\|\partial_2 h(\cdot,\cdot,x_3)\|_2^{\frac{1}{2}}\,\|g(\cdot,\cdot,x_3)\|_2^{\frac{1}{2}}\,\|\partial_1 g(\cdot,\cdot,x_3)\|_2^{\frac{1}{2}}\,\|f(\cdot,\cdot,x_3)\|_2\,dx_3\\ &\leq 2\sup_{x_3\in\mathbb{R}}\left\{\|h(\cdot,\cdot,x_3)\|_2^{\frac{1}{2}}\|g(\cdot,\cdot,x_3)\|_2^{\frac{1}{2}}\right\}\int_{\mathbb{R}} \|\partial_2 h(\cdot,\cdot,x_3)\|_2^{\frac{1}{2}}\,\|\partial_1 g(\cdot,\cdot,x_3)\|_2^{\frac{1}{2}}\,\|f(\cdot,\cdot,x_3)\|_2\,dx_3\\ &\leq 2\sup_{x_3\in\mathbb{R}}\left\{\|h(\cdot,\cdot,x_3)\|_2^{\frac{1}{2}}\|g(\cdot,\cdot,x_3)\|_2^{\frac{1}{2}}\right\}\|\partial_2 h\|_2^{\frac{1}{2}}\,\|\partial_1 g\|_2^{\frac{1}{2}}\,\|f\|_2.\end{aligned}$$

Por fim, note que novamente pelo Teorema Fundamental do Cálculo e pela Desigualdade de Hölder, chegamos a

$$\begin{aligned}|h(x)|^2 &= 2\int_{-\infty}^{z} h(x_1,x_2,x_3)\partial_3 h(x_1,x_2,x_3)\,dx_3\\ &\leq 2\left(\int_{\mathbb{R}} |h(x_1,x_2,x_3)|^2\,dx_3\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} |\partial_3 h(x_1,x_2,x_3)|^2\;dx_3\right)^{\frac{1}{2}}.\end{aligned}$$

Desta forma, podemos escrever

$$\begin{aligned}\|h(\cdot,\cdot,x_3)\|_2^2 &\leq 2\int_{\mathbb{R}^2} \|h(x_1,x_2,\cdot)\|_2 \left\|\partial_3 h(x_1,x_2,\cdot)\right\|_2 \, dx_1dx_2 \\ &\leq 2\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \|h(x_1,x_2,\cdot)\|_2^2 \, dx_2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \|\partial_3 h(x_1,x_2,\cdot)\|_2^2 \, dx_2\right)^{\frac{1}{2}} dx_1 \\ &\leq 2\|h\|_2 \left\|\partial_3 h\right\|_2 .\end{aligned}$$

Consequentemente,

$$\|h(\cdot,\cdot,x_3)\|_2^{\frac{1}{2}} \leq \sqrt[4]{2}\|h\|_2^{\frac{1}{4}} \left\|\partial_3 h\right\|_2^{\frac{1}{4}} .$$

Analogamente, obtemos

$$\|g(\cdot,\cdot,x_3)\|_2^{\frac{1}{2}} \leq \sqrt[4]{2}\|g\|_2^{\frac{1}{4}} \left\|\partial_3 g\right\|_2^{\frac{1}{4}} .$$

Logo,

$$\sup_{x_3\in\mathbb{R}} \{\|h(\cdot,\cdot,x_3)\|_2^{\frac{1}{2}} \|g(\cdot,\cdot,x_3)\|_2^{\frac{1}{2}}\} \leq \sqrt{2}\|g\|_2^{\frac{1}{4}} \left\|\partial_3 g\right\|_2^{\frac{1}{4}} \|h\|_2^{\frac{1}{4}} \left\|\partial_3 h\right\|_2^{\frac{1}{4}} .$$

Portanto,

$$\int_{\mathbb{R}^3} |fgh| \, dx \leq 2\sqrt{2}\|f\|_2 \left\|\partial_1 g\right\|_2^{\frac{1}{2}} \left\|\partial_2 h\right\|_2^{\frac{1}{2}} \|g\|_2^{\frac{1}{4}} \|h\|_2^{\frac{1}{4}} \left\|\partial_3 g\right\|_2^{\frac{1}{4}} \left\|\partial_3 h\right\|_2^{\frac{1}{4}} .$$

Isto completa a prova do lema em questão. □

Agora, estamos preparados para apresentar o resultado principal desta seção.

Teorema 3.1. *Sejam* $(\mathbf{u_0}, \mathbf{w_0}, \mathbf{b_0}) \in L^2(\mathbb{R}^3)$ *com* $\nabla \cdot \mathbf{u_0} = \nabla \cdot \mathbf{b_0} = 0$. *Assuma que* $T > 0$. *Considere que* $(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t)$ *é uma solução fraca das equações magneto-micropolares* (1) *em* $[0, T]$ *com condição inicial* $(\mathbf{u_0}, \mathbf{w_0}, \mathbf{b_0})$. *Se*

$$(\nabla u_3, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})(\cdot, t) \in L^{\frac{32}{7}}(0, T; L^2(\mathbb{R}^3)), \tag{3.3}$$

então $(\mathbf{u}, \mathbf{w}, \mathbf{b})$ *é suave em* $\mathbb{R}^3 \times (0, T)$.

Demonstração. Seja $0 < \epsilon < T$ arbitrário. Como $(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t) \in L^2(0, T; H^1(\mathbb{R}^3))$, conclui-se que $(\nabla\mathbf{u}, \nabla\mathbf{w}, \nabla\mathbf{b})(\cdot, t) \in L^2(0, T; L^2(\mathbb{R}^3))$. Isto nos informa que existe $0 < t_0 < \epsilon$ tal que

$(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\cdot,t_0)\in L^2(\mathbb{R}^3)$; caso contrário,

$$\infty=\int_0^{\epsilon}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\cdot,t)\|_2^2\,dt\le\int_0^{T}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\cdot,t)\|_2^2\,dt<\infty.$$

Sabemos que existe uma única solução forte $(\mathbf{u}',\mathbf{w}',\mathbf{b}')(\cdot,t)\in C([t_0,T^*);H^1(\mathbb{R}^3))\cap L^2(t_0,T^*;H^2(\mathbb{R}^3))$ de (1) (sobre solução forte ver [46, 53] e referências inclusas) tal que $(\mathbf{u}',\mathbf{w}',\mathbf{b}')(\cdot,t_0)=(\mathbf{u},\mathbf{w},\mathbf{b})(\cdot,t_0)$, $t=T^*$ é o tempo máximo de existência desta solução e $(\mathbf{u}',\mathbf{w}',\mathbf{b}')\in C^\infty(\mathbb{R}^3\times(0,T^*))$ (pois consideramos $\epsilon>0$ arbitrário). Assim, se $T\le T^*$, então $(\mathbf{u},\mathbf{w},\mathbf{b})(\cdot,t)(=(\mathbf{u}',\mathbf{w}',\mathbf{b}')(\cdot,t))$ é suave em $\mathbb{R}^3\times(0,T)$. Em contrapartida, assumindo $T^*<T$, provaremos abaixo que $\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\cdot,t)\|_2$ é limitada quando $t\nearrow T^*$. Porém, isto não pode ocorrer desde que, deste modo, $(\mathbf{u}',\mathbf{w}',\mathbf{b}')$ poderia ser estendida além de $t=T^*$.

Inicialmente, iremos encontrar uma estimativa para a norma-L^2 de $(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})$. Como podemos escrever a derivada de tal norma da forma

$$\frac{1}{2}\frac{d}{dt}\|(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})\|_2^2=\frac{1}{2}\frac{d}{dt}\|\nabla_h\mathbf{u}\|_2^2+\frac{1}{2}\frac{d}{dt}\|\nabla_h\mathbf{w}\|_2^2+\frac{1}{2}\frac{d}{dt}\|\nabla_h\mathbf{b}\|_2^2, \tag{3.4}$$

então vamos em busca de estimativas para cada uma das parcelas do lado direito da igualdade acima, e dessa forma conseguiremos estimar $\|(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})\|_2$. Assim, observe que pela primeira equação do sistema (1), obtemos

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla_h\mathbf{u}\|_2^2&=(\nabla_h\partial_t\mathbf{u},\nabla_h\mathbf{u})_2=\sum_{i=1}^{2}(\partial_i\partial_t\mathbf{u},\partial_i\mathbf{u})_2\\&=-\sum_{i=1}^{2}(\partial_t\mathbf{u},\partial_i^2\mathbf{u})_2=-(\partial_t\mathbf{u},\Delta_h\mathbf{u})_2\\&=-(\mu+\chi)(\Delta\mathbf{u},\Delta_h\mathbf{u})_2+(\mathbf{u}\cdot\nabla\mathbf{u},\Delta_h\mathbf{u})_2-(\mathbf{b}\cdot\nabla\mathbf{b},\Delta_h\mathbf{u})_2\\&\quad+(\nabla(p+\frac{1}{2}|\mathbf{b}|^2),\Delta_h\mathbf{u})_2-\chi(\nabla\times\mathbf{w},\Delta_h\mathbf{u})_2.\end{aligned}$$

Mas, por integração por partes, obtemos

$$\begin{aligned}(\Delta\mathbf{u},\Delta_h\mathbf{u})_2&=\sum_{i=1}^{3}\sum_{j=1}^{2}(\partial_i^2\mathbf{u},\partial_j^2\mathbf{u})_2=-\sum_{i=1}^{3}\sum_{j=1}^{2}(\partial_i\mathbf{u},\partial_i\partial_j^2\mathbf{u})_2\\&=\sum_{i=1}^{3}\sum_{j=1}^{2}(\partial_j\partial_i\mathbf{u},\partial_i\partial_j\mathbf{u})_2=\|\nabla\nabla_h\mathbf{u}\|_2^2\end{aligned} \tag{3.5}$$

e, além disso

$$
\begin{aligned}
(\nabla(p+\frac{1}{2}|\mathbf{b}|^2), \Delta_h \mathbf{u})_2 &= \sum_{i=1}^{3}(\partial_i(p+\frac{1}{2}|\mathbf{b}|^2), \Delta_h u_i)_2 \\
&= -\sum_{i=1}^{3}(p+\frac{1}{2}|\mathbf{b}|^2, \Delta_h \partial_i u_i)_2 \\
&= 0,
\end{aligned}
$$

pois $\nabla \cdot \mathbf{u} = 0$. Logo,

$$
\frac{1}{2}\frac{d}{dt}\|\nabla_h \mathbf{u}\|_2^2 + (\mu+\chi)\|\nabla\nabla_h \mathbf{u}\|_2^2 = (\mathbf{u}\cdot\nabla\mathbf{u}, \Delta_h \mathbf{u})_2 - (\mathbf{b}\cdot\nabla\mathbf{b}, \Delta_h \mathbf{u})_2 - \chi(\nabla\times\mathbf{w}, \Delta_h \mathbf{u})_2. \tag{3.6}
$$

Agora, veja que pela segunda equação do sistema (1), temos

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla_h \mathbf{w}\|_2^2 &= (\nabla_h \partial_t \mathbf{w}, \nabla_h \mathbf{w})_2 \\
&= -(\partial_t \mathbf{w}, \Delta_h \mathbf{w})_2 \\
&= -\gamma(\Delta \mathbf{w}, \Delta_h \mathbf{w})_2 - \kappa(\nabla(\nabla\cdot\mathbf{w}), \Delta_h \mathbf{w})_2 + 2\chi(\mathbf{w}, \Delta_h \mathbf{w})_2 + (\mathbf{u}\cdot\nabla\mathbf{w}, \Delta_h \mathbf{w})_2 \\
&\quad - \chi(\nabla\times\mathbf{u}, \Delta_h \mathbf{w})_2.
\end{aligned}
$$

Permita-nos avaliar alguns dos termos do lado direito das igualdades aima. Asim sendo, por integração por partes, obtemos

$$
\begin{aligned}
(\nabla(\nabla\cdot\mathbf{w}), \Delta_h \mathbf{w})_2 &= \sum_{i=1}^{3}(\partial_i(\nabla\cdot\mathbf{w}), \Delta_h w_i)_2 = \sum_{i=1}^{3}\sum_{j=1}^{2}(\partial_i(\nabla\cdot\mathbf{w}), \partial_j^2 w_i)_2 \\
&= -\sum_{i=1}^{3}\sum_{j=1}^{2}(\partial_j\partial_i(\nabla\cdot\mathbf{w}), \partial_j w_i)_2 = \sum_{i=1}^{3}\sum_{j=1}^{2}(\partial_j(\nabla\cdot\mathbf{w}), \partial_j\partial_i w_i)_2 \\
&= \sum_{j=1}^{2}(\partial_j(\nabla\cdot\mathbf{w}), \partial_j(\nabla\cdot\mathbf{w}))_2 = \|\nabla_h(\nabla\cdot\mathbf{w})\|_2^2.
\end{aligned} \tag{3.7}
$$

Além disso,

$$
(\mathbf{w}, \Delta_h \mathbf{w})_2 \quad = \quad \sum_{j=1}^{2}(\mathbf{w}, \partial_j^2 \mathbf{w})_2 = -\sum_{j=1}^{2}(\partial_j \mathbf{w}, \partial_j \mathbf{w})_2 = -\|\nabla_h \mathbf{w}\|_2^2.
$$

Analogamente a (3.5), temos que

$$
(\Delta \mathbf{w}, \Delta_h \mathbf{w})_2 = \|\nabla\nabla_h \mathbf{w}\|_2^2.
$$

Desta forma, obtemos

$$\frac{1}{2}\frac{d}{dt}\|\nabla_h \mathbf{w}\|_2^2 + \gamma\|\nabla\nabla_h \mathbf{w}\|_2^2 + \kappa\|\nabla_h(\nabla\cdot\mathbf{w})\|_2^2 + 2\chi\|\nabla_h \mathbf{w}\|_2^2 = (\mathbf{u}\cdot\nabla\mathbf{w}, \Delta_h\mathbf{w})_2 - \chi(\nabla\times\mathbf{u}, \Delta_h\mathbf{w})_2. \tag{3.8}$$

E, veja ainda que pela terceira equação do sistema (1), podemos escrever

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla_h \mathbf{b}\|_2^2 &= (\nabla_h\partial_t\mathbf{b}, \nabla_h\mathbf{b})_2\\ &= -(\partial_t\mathbf{b}, \Delta_h\mathbf{b})_2\\ &= -\nu(\Delta\mathbf{b}, \Delta_h\mathbf{b})_2 + (\mathbf{u}\cdot\nabla\mathbf{b}, \Delta_h\mathbf{b})_2 - (\mathbf{b}\cdot\nabla\mathbf{u}, \Delta_h\mathbf{b})_2,\end{aligned}$$

ou seja,

$$\frac{1}{2}\frac{d}{dt}\|\nabla_h \mathbf{b}\|_2^2 + \nu\|\nabla\nabla_h\mathbf{b}\|_2^2 = (\mathbf{u}\cdot\nabla\mathbf{b}, \Delta_h\mathbf{b})_2 - (\mathbf{b}\cdot\nabla\mathbf{u}, \Delta_h\mathbf{b})_2. \tag{3.9}$$

Desta forma, substituindo (3.6), (3.8) e (3.9) em (3.4), encontramos

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(\nabla_h\mathbf{u}, \nabla_h\mathbf{w}, \nabla_h\mathbf{b})\|_2^2 + (\mu+\chi)\|\nabla\nabla_h\mathbf{u}\|_2^2 + \gamma\|\nabla\nabla_h\mathbf{w}\|_2^2 + \nu\|\nabla\nabla_h\mathbf{b}\|_2^2 + \kappa\|\nabla_h(\nabla\cdot\mathbf{w})\|_2^2\\ &+ 2\chi\|\nabla_h\mathbf{w}\|_2^2 = (\mathbf{u}\cdot\nabla\mathbf{u}, \Delta_h\mathbf{u})_2 - (\mathbf{b}\cdot\nabla\mathbf{b}, \Delta_h\mathbf{u})_2 - \chi(\nabla\times\mathbf{w}, \Delta_h\mathbf{u})_2 + (\mathbf{u}\cdot\nabla\mathbf{w}, \Delta_h\mathbf{w})_2\\ &- \chi(\nabla\times\mathbf{u}, \Delta_h\mathbf{w})_2 + (\mathbf{u}\cdot\nabla\mathbf{b}, \Delta_h\mathbf{b})_2 - (\mathbf{b}\cdot\nabla\mathbf{u}, \Delta_h\mathbf{b})_2.\end{aligned} \tag{3.10}$$

Neste momento, vamos em busca de relações para todas as parcelas do lado direito da igualdade acima. Primeiramente, observe que

$$\begin{aligned}(\mathbf{u}\cdot\nabla\mathbf{u}, \Delta_h\mathbf{u})_2 &= \sum_{i,j=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_i u_j)\Delta_h u_j\,dx\\ &= \sum_{i,j=1}^{2}\int_{\mathbb{R}^3} u_i(\partial_i u_j)\Delta_h u_j\,dx + \sum_{j=1}^{2}\int_{\mathbb{R}^3} u_3(\partial_3 u_j)\Delta_h u_j\,dx\\ &\quad + \sum_{i=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_i u_3)\Delta_h u_3\,dx\\ &=: M_1 + M_2 + M_3.\end{aligned} \tag{3.11}$$

A partir de agora, buscaremos estimativas para $M_j (j = 1, 2, 3)$. Primeiramente vamos provar a seguinte afirmação:

Afirmação 3.1. *A seguinte igualdade é válida:*

$$\sum_{i,j=1}^{2}\int_{\mathbb{R}^3} u_i(\partial_i u_j)\Delta_h u_j\,dx = \frac{1}{2}\sum_{i,j=1}^{2}\int_{\mathbb{R}^3}(\partial_i u_j)^2(\partial_3 u_3)\,dx - \int_{\mathbb{R}^3}(\partial_1 u_1)(\partial_2 u_2)(\partial_3 u_3)\,dx$$
$$+\int_{\mathbb{R}^3}(\partial_1 u_2)(\partial_2 u_1)(\partial_3 u_3)\,dx.$$

Com efeito, observe que, por integração por partes, podemos escrever

$$\sum_{i,j=1}^{2}\int_{\mathbb{R}^3} u_i(\partial_i u_j)\Delta_h u_j\,dx = \sum_{i,j,k=1}^{2}\int_{\mathbb{R}^3} u_i(\partial_i u_j)(\partial_k^2 u_j)\,dx$$
$$= -\sum_{i,j,k=1}^{2}\int_{\mathbb{R}^3}(\partial_k u_i)(\partial_i u_j)(\partial_k u_j)\,dx - \sum_{i,j,k=1}^{2}\int_{\mathbb{R}^3} u_i(\partial_k\partial_i u_j)(\partial_k u_j)\,dx.$$

Mas, por integração por partes novamente, encontramos

$$\sum_{i,j,k=1}^{2}\int_{\mathbb{R}^3} u_i(\partial_k\partial_i u_j)(\partial_k u_j)\,dx = -\sum_{i,j,k=1}^{2}\int_{\mathbb{R}^3}(\partial_i u_i)(\partial_k u_j)^2\,dx - \sum_{i,j,k=1}^{2}\int_{\mathbb{R}^3} u_i(\partial_i\partial_k u_j)(\partial_k u_j)\,dx,$$

ou equivalentemente,

$$\sum_{i,j,k=1}^{2}\int_{\mathbb{R}^3} u_i(\partial_k\partial_i u_j)(\partial_k u_j)\,dx = -\frac{1}{2}\sum_{i,j,k=1}^{2}\int_{\mathbb{R}^3}(\partial_i u_i)(\partial_k u_j)^2\,dx.$$

Assim sendo, conclui-se

$$\sum_{i,j=1}^{2}\int_{\mathbb{R}^3} u_i(\partial_i u_j)\Delta_h u_j\,dx = -\sum_{i,j,k=1}^{2}\int_{\mathbb{R}^3}(\partial_k u_i)(\partial_i u_j)(\partial_k u_j)\,dx + \frac{1}{2}\sum_{i,j,k=1}^{2}\int_{\mathbb{R}^3}(\partial_i u_i)(\partial_k u_j)^2\,dx.$$

Como $\mathbf{u}$ é livre de divergente, então

$$\frac{1}{2}\sum_{i,j,k=1}^{2}\int_{\mathbb{R}^3}(\partial_i u_i)(\partial_k u_j)^2\,dx = -\frac{1}{2}\sum_{j,k=1}^{2}\int_{\mathbb{R}^3}(\partial_3 u_3)(\partial_k u_j)^2\,dx.$$

Agora note que

$$
\begin{aligned}
&-\sum_{i,j,k=1}^{2}\int_{\mathbb{R}^3}(\partial_k u_i)(\partial_i u_j)(\partial_k u_j)\,dx = -\int_{\mathbb{R}^3}(\partial_1 u_1)^3\,dx - \int_{\mathbb{R}^3}(\partial_1 u_1)^2(\partial_2 u_2)\,dx - \int_{\mathbb{R}^3}(\partial_2 u_2)^3\,dx\\
&-\int_{\mathbb{R}^3}(\partial_2 u_2)^2(\partial_1 u_1)\,dx - \int_{\mathbb{R}^3}(\partial_1 u_1)(\partial_1 u_2)^2\,dx - \int_{\mathbb{R}^3}(\partial_1 u_2)^2(\partial_2 u_2)\,dx - \int_{\mathbb{R}^3}(\partial_2 u_1)^2(\partial_1 u_1)\,dx\\
&-\int_{\mathbb{R}^3}(\partial_2 u_2)(\partial_2 u_1)^2\,dx + \int_{\mathbb{R}^3}(\partial_1 u_1)^2(\partial_2 u_2)\,dx + \int_{\mathbb{R}^3}(\partial_2 u_2)^2(\partial_1 u_1)\,dx - \int_{\mathbb{R}^3}(\partial_1 u_2)(\partial_2 u_1)(\partial_1 u_1)\,dx\\
&-\int_{\mathbb{R}^3}(\partial_2 u_1)(\partial_1 u_2)(\partial_2 u_2)\,dx.
\end{aligned}
$$

Consequentemente, chegamos a

$$
\begin{aligned}
-\sum_{i,j,k=1}^{2}\int_{\mathbb{R}^3}(\partial_k u_i)(\partial_i u_j)(\partial_k u_j)\,dx = &\sum_{j,k=1}^{2}\int_{\mathbb{R}^3}(\partial_3 u_3)(\partial_k u_j)^2\,dx - \int_{\mathbb{R}^3}(\partial_3 u_3)(\partial_1 u_1)(\partial_2 u_2)\,dx\\
&+\int_{\mathbb{R}^3}(\partial_3 u_3)(\partial_1 u_2)(\partial_2 u_1)\,dx.
\end{aligned}
$$

Portanto,

$$
\begin{aligned}
\sum_{i,j=1}^{2}\int_{\mathbb{R}^3}u_i(\partial_i u_j)\Delta_h u_j\,dx = &\frac{1}{2}\sum_{i,j,k=1}^{2}\int_{\mathbb{R}^3}(\partial_3 u_3)(\partial_k u_j)^2\,dx - \int_{\mathbb{R}^3}(\partial_3 u_3)(\partial_1 u_1)(\partial_2 u_2)\,dx\\
&+\int_{\mathbb{R}^3}(\partial_3 u_3)(\partial_1 u_2)(\partial_2 u_1)\,dx.
\end{aligned}
$$

Isto finaliza a prova da Afirmação 3.1.

Note que, pela afirmação acima, temos que

$$
M_1 = \frac{1}{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}(\partial_3 u_3)(\partial_k u_j)^2\,dx - \int_{\mathbb{R}^3}(\partial_1 u_1)(\partial_2 u_2)(\partial_3 u_3)\,dx + \int_{\mathbb{R}^3}(\partial_2 u_1)(\partial_1 u_2)(\partial_3 u_3)\,dx. \tag{3.12}
$$

Para M_2 e M_3 usaremos a integração por partes em ordem a obter

$$
\begin{aligned}
M_2 &:= \sum_{j=1}^{2}\int_{\mathbb{R}^3}u_3(\partial_3 u_j)\Delta_h u_j\,dx\\
&= \sum_{j,k=1}^{2}\int_{\mathbb{R}^3}u_3(\partial_3 u_j)(\partial_k^2 u_j)\,dx\\
&= -\sum_{j,k=1}^{2}\int_{\mathbb{R}^3}(\partial_k u_3)(\partial_3 u_j)(\partial_k u_j)\,dx - \sum_{j,k=1}^{2}\int_{\mathbb{R}^3}u_3(\partial_k\partial_3 u_j)(\partial_k u_j)\,dx.
\end{aligned}
$$

Por outro lado, observe que

$$\sum_{j,k=1}^{2}\int_{\mathbb{R}^3} u_3(\partial_k\partial_3 u_j)(\partial_k u_j)\,dx = -\sum_{j,k=1}^{2}\int_{\mathbb{R}^3}(\partial_3 u_3)(\partial_k u_j)^2\,dx - \sum_{j,k=1}^{2}\int_{\mathbb{R}^3} u_3(\partial_k\partial_3 u_j)(\partial_k u_j)\,dx.$$

Assim, podemos inferir

$$\sum_{j,k=1}^{2}\int_{\mathbb{R}^3} u_3(\partial_k\partial_3 u_j)(\partial_k u_j)\,dx = -\frac{1}{2}\sum_{j,k=1}^{2}\int_{\mathbb{R}^3}(\partial_3 u_3)(\partial_k u_j)^2\,dx.$$

Logo,

$$M_2 = -\sum_{j,k=1}^{2}\int_{\mathbb{R}^3}(\partial_k u_3)(\partial_3 u_j)(\partial_k u_j)\,dx + \frac{1}{2}\sum_{j,k=1}^{2}\int_{\mathbb{R}^3}(\partial_3 u_3)(\partial_k u_j)^2\,dx. \tag{3.13}$$

Além disso, é fácil concluir que

$$\begin{aligned} M_3 &:= \sum_{i=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_i u_3)\Delta_h u_3\,dx \\ &= \sum_{i=1}^{3}\sum_{j=1}^{2}\int_{\mathbb{R}^3} u_i(\partial_i u_3)(\partial_j^2 u_3)\,dx \\ &= -\sum_{i=1}^{3}\sum_{j=1}^{2}\int_{\mathbb{R}^3}(\partial_j u_i)(\partial_i u_3)(\partial_j u_3)\,dx - \sum_{i=1}^{3}\sum_{j=1}^{2}\int_{\mathbb{R}^3} u_i(\partial_j\partial_i u_3)(\partial_j u_3)\,dx. \end{aligned}$$

Por outro lado, pode-se inferir

$$\sum_{i=1}^{3}\sum_{j=1}^{2}\int_{\mathbb{R}^3} u_i(\partial_j\partial_i u_3)(\partial_j u_3)\,dx = -\sum_{i=1}^{3}\sum_{j=1}^{2}\int_{\mathbb{R}^3}(\partial_i u_i)(\partial_j u_3)^2\,dx - \sum_{i=1}^{3}\sum_{j=1}^{2}\int_{\mathbb{R}^3} u_i(\partial_j\partial_i u_3)(\partial_j u_3)\,dx.$$

Com isso,

$$\sum_{i=1}^{3}\sum_{j=1}^{2}\int_{\mathbb{R}^3} u_i(\partial_j\partial_i u_3)(\partial_j u_3)\,dx = 0,$$

pois $\mathbf{u}$ é livre de divergente. Logo,

$$M_3 = -\sum_{i=1}^{3}\sum_{j=1}^{2}\int_{\mathbb{R}^3}(\partial_j u_i)(\partial_i u_3)(\partial_j u_3)\,dx. \tag{3.14}$$

Substituindo (3.12), (3.13) e (3.14) em (3.11), encontramos

$$
\begin{aligned}
(\mathbf{u}\cdot\nabla\mathbf{u},\Delta_h\mathbf{u})_2 &= \sum_{j,k=1}^{2}\int_{\mathbb{R}^3}(\partial_3 u_3)(\partial_k u_j)^2\,dx - \sum_{j,k=1}^{2}\int_{\mathbb{R}^3}(\partial_k u_3)(\partial_3 u_j)(\partial_k u_j)\,dx\\
&\quad -\sum_{i=1}^{3}\sum_{j=1}^{2}\int_{\mathbb{R}^3}(\partial_j u_i)(\partial_i u_3)(\partial_j u_3)\,dx - \int_{\mathbb{R}^3}(\partial_1 u_1)(\partial_2 u_2)(\partial_3 u_3)\,dx\\
&\quad +\int_{\mathbb{R}^3}(\partial_2 u_1)(\partial_1 u_2)(\partial_3 u_3)\,dx\\
&\le \sum_{j,k=1}^{2}\int_{\mathbb{R}^3}|\partial_3 u_3||\partial_k u_j|^2\,dx + \sum_{j,k=1}^{2}\int_{\mathbb{R}^3}|\partial_k u_3||\partial_3 u_j||\partial_k u_j|\,dx\\
&\quad +\sum_{i=1}^{3}\sum_{j=1}^{2}\int_{\mathbb{R}^3}|\partial_j u_i||\partial_i u_3||\partial_j u_3|\,dx + \int_{\mathbb{R}^3}|\partial_1 u_1||\partial_2 u_2||\partial_3 u_3|\,dx\\
&\quad +\int_{\mathbb{R}^3}|\partial_2 u_1||\partial_1 u_2||\partial_3 u_3|\,dx.
\end{aligned}
$$

Por conseguinte,

$$
\begin{aligned}
(\mathbf{u}\cdot\nabla\mathbf{u},\Delta_h\mathbf{u})_2 &\le C\int_{\mathbb{R}^3}|\nabla u_3||\nabla_h\mathbf{u}|^2\,dx + C\int_{\mathbb{R}^3}|\nabla_h u_3||\nabla\mathbf{u}||\nabla_h\mathbf{u}|\,dx\\
&\quad + C\int_{\mathbb{R}^3}|\nabla_h u_3||\nabla u_3||\nabla_h\mathbf{u}|\,dx\\
&\le C\int_{\mathbb{R}^3}|\nabla u_3||\nabla\mathbf{u}||\nabla_h\mathbf{u}|\,dx.
\end{aligned}
$$

Agora, observe que

$$
\begin{aligned}
-(\mathbf{b}\cdot\nabla\mathbf{b},\Delta_h\mathbf{u})_2 &= -\sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}b_i(\partial_i b_j)(\partial_k^2 u_j)\,dx\\
&= \sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}(\partial_k b_i)(\partial_i b_j)(\partial_k u_j)\,dx + \sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}b_i(\partial_k\partial_i b_j)(\partial_k u_j)\,dx\\
&\le C\int_{\mathbb{R}^3}|\nabla_h\mathbf{b}||\nabla\mathbf{b}||\nabla_h\mathbf{u}|\,dx + \sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}b_i(\partial_k\partial_i b_j)(\partial_k u_j)\,dx.
\end{aligned}
$$

Além disso, podemos escrever

$$
\begin{aligned}
(\mathbf{u}\cdot\nabla\mathbf{w},\Delta_h\mathbf{w})_2 &= \sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}u_i(\partial_i w_j)(\partial_k^2 w_j)\,dx\\
&= -\sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}(\partial_k u_i)(\partial_i w_j)(\partial_k w_j)\,dx - \sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}u_i(\partial_k\partial_i w_j)(\partial_k w_j)\,dx.
\end{aligned}
$$

Mas, como $\mathbf{u}$ é livre de divergente, temos

$$
\begin{aligned}
-\sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} u_i(\partial_k\partial_i w_j)(\partial_k w_j)\,dx &= \sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} (\partial_i u_i)(\partial_k w_j)^2\,dx \\
&\quad + \sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} u_i(\partial_i\partial_k w_j)(\partial_k w_j)\,dx \\
&= \sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} u_i(\partial_i\partial_k w_j)(\partial_k w_j)\,dx.
\end{aligned}
$$

Consequentemente, infere-se

$$
\sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} u_i(\partial_k\partial_i w_j)(\partial_k w_j)\,dx = 0.
$$

Assim, é verdade que

$$
(\mathbf{u}\cdot\nabla\mathbf{w},\Delta_h\mathbf{w})_2 = -\sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} (\partial_k u_i)(\partial_i w_j)(\partial_k w_j)\,dx \le C\int_{\mathbb{R}^3} |\nabla_h\mathbf{w}||\nabla\mathbf{w}||\nabla_h\mathbf{u}|\,dx.
$$

Analogamente, conseguimos a seguinte desigualdade:

$$
(\mathbf{u}\cdot\nabla\mathbf{b},\Delta_h\mathbf{b})_2 \le C\int_{\mathbb{R}^3} |\nabla_h\mathbf{b}||\nabla\mathbf{b}||\nabla_h\mathbf{u}|\,dx,
$$

pois $\nabla\cdot\mathbf{u}=0$. Observe ainda que,

$$
\begin{aligned}
-(\mathbf{b}\cdot\nabla\mathbf{u},\Delta_h\mathbf{b})_2 &= -\sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} b_i(\partial_i u_j)(\partial_k^2 b_j)\,dx \\
&= \sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} (\partial_k b_i)(\partial_i u_j)(\partial_k b_j)\,dx + \sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} b_i(\partial_k\partial_i u_j)(\partial_k b_j)\,dx.
\end{aligned}
$$

Porém, a última integral acima pode ser trabalhada da seguinte forma:

$$
\begin{aligned}
\sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} b_i(\partial_k\partial_i u_j)(\partial_k b_j)\,dx &= -\sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} (\partial_i b_i)(\partial_k b_j)(\partial_k u_j)\,dx \\
&\quad - \sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} b_i(\partial_i\partial_k b_j)(\partial_k u_j)\,dx \\
&= -\sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} b_i(\partial_i\partial_k b_j)(\partial_k u_j)\,dx,
\end{aligned}
$$

já que $\nabla \cdot \mathbf{b} = 0$. Assim sendo, chegamos a

$$\begin{aligned}
-(\mathbf{b} \cdot \nabla \mathbf{u}, \Delta_h \mathbf{b})_2 &= \sum_{i,j=1}^{3} \sum_{k=1}^{2} \int_{\mathbb{R}^3} (\partial_k b_i)(\partial_i u_j)(\partial_k b_j)\, dx - \sum_{i,j=1}^{3} \sum_{k=1}^{2} \int_{\mathbb{R}^3} b_i(\partial_i \partial_k b_j)(\partial_k u_j)\, dx \\
&\leq C \int_{\mathbb{R}^3} |\nabla_h \mathbf{b}||\nabla \mathbf{u}||\nabla_h \mathbf{b}|\, dx - \sum_{i,j=1}^{3} \sum_{k=1}^{2} \int_{\mathbb{R}^3} b_i(\partial_i \partial_k b_j)(\partial_k u_j)\, dx.
\end{aligned}$$

Analogamente ao feito em (2.54), temos que

$$-\chi(\nabla \times \mathbf{w}, \Delta_h \mathbf{u})_2 - \chi(\nabla \times \mathbf{u}, \Delta_h \mathbf{w})_2 = 2\chi(\nabla \times (\nabla_h \mathbf{u}), \nabla_h \mathbf{w})_2 \leq \chi \|\nabla_h \nabla \mathbf{u}\|_2^2 + \chi \|\nabla_h \mathbf{w}\|_2^2. \tag{3.15}$$

Desta forma, voltando para (3.10), temos

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})\|_2^2 + \mu\|\nabla\nabla_h \mathbf{u}\|_2^2 + \gamma\|\nabla\nabla_h \mathbf{w}\|_2^2 + \nu\|\nabla\nabla_h \mathbf{b}\|_2^2 + \kappa\|\nabla_h(\nabla \cdot \mathbf{w})\|_2^2 + \chi\|\nabla_h \mathbf{w}\|_2^2 \\
&\leq C \int_{\mathbb{R}^3} |\nabla u_3|\ |\nabla \mathbf{u}|\ |\nabla_h \mathbf{u}|\, dx + C \int_{\mathbb{R}^3} |\nabla_h \mathbf{w}|\ |\nabla \mathbf{w}|\ |\nabla_h \mathbf{u}|\, dx + C \int_{\mathbb{R}^3} |\nabla_h \mathbf{b}|\ |\nabla \mathbf{b}|\ |\nabla_h \mathbf{u}|\, dx \\
&\quad + C \int_{\mathbb{R}^3} |\nabla_h \mathbf{b}|\ |\nabla \mathbf{u}|\ |\nabla_h \mathbf{b}|\, dx \\
&\leq C \int_{\mathbb{R}^3} |(\nabla u_3, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})|\ |(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})|\ |(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})|\, dx.
\end{aligned}$$

Logo, pelo Lema 3.1, obtemos

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})\|_2^2 + \mu\|\nabla\nabla_h \mathbf{u}\|_2^2 + \gamma\|\nabla\nabla_h \mathbf{w}\|_2^2 + \nu\|\nabla\nabla_h \mathbf{b}\|_2^2 + \kappa\|\nabla_h(\nabla \cdot \mathbf{w})\|_2^2 + \chi\|\nabla_h \mathbf{w}\|_2^2 \\
&\leq C \int_{\mathbb{R}^3} |(\nabla u_3, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})|\ |(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})|\ |(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})|\, dx \\
&\leq C\|(\nabla u_3, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})\|_2\ \|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})\|_2^{\frac{1}{4}}\ \|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})\|_2^{\frac{1}{4}} \|(\nabla\nabla_h \mathbf{u}, \nabla\nabla_h \mathbf{w}, \nabla\nabla_h \mathbf{b})\|_2^{\frac{3}{2}}.
\end{aligned}$$

Pela Desigualdade de Young, $p = 4$ e $q = \frac{4}{3}$, obtemos

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})\|_2^2 + \alpha\|(\nabla\nabla_h \mathbf{u}, \nabla\nabla_h \mathbf{w}, \nabla\nabla_h \mathbf{b})\|_2^2 + \kappa\|\nabla_h(\nabla \cdot \mathbf{w})\|_2^2 + \chi\|\nabla_h \mathbf{w}\|_2^2 \\
&\leq C\|(\nabla u_3, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})\|_2^4\ \|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})\|_2\ \|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})\|_2 \\
&\quad + \frac{\alpha}{2}\|(\nabla\nabla_h \mathbf{u}, \nabla\nabla_h \mathbf{w}, \nabla\nabla_h \mathbf{b})\|_2^2,
\end{aligned}$$

onde $\alpha = \min\{\mu, \gamma, \nu\}$. Assim, chegamos a

$$\begin{aligned}
&\frac{d}{dt}\|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})\|_2^2 + \alpha\|(\nabla\nabla_h \mathbf{u}, \nabla\nabla_h \mathbf{w}, \nabla\nabla_h \mathbf{b})\|_2^2 + 2\kappa\|\nabla_h(\nabla\cdot\mathbf{w})\|_2^2 + 2\chi\|\nabla_h \mathbf{w}\|_2^2 \\
&\le C\|(\nabla u_3, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})\|_2^4 \; \|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})\|_2 \; \|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})\|_2.
\end{aligned}$$

Integrando a desigualdade acima sobre $[T^* - \tau, t]$, onde τ será escolhido mais adiante, obtemos

$$\begin{aligned}
&\|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})(\cdot, t)\|_2^2 + \alpha\int_{T^*-\tau}^{t} \|(\nabla\nabla_h \mathbf{u}, \nabla\nabla_h \mathbf{w}, \nabla\nabla_h \mathbf{b})(\cdot, s)\|_2^2\, ds + 2\kappa\int_{T^*-\tau}^{t} \|\nabla_h\nabla\cdot\mathbf{w}(\cdot, s)\|_2^2\, ds \\
&+ 2\chi\int_{T^*-\tau}^{t} \|\nabla_h \mathbf{w}(\cdot, s)\|_2^2\, ds \le C + C\int_{T^*-\tau}^{t} \|(\nabla u_3, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})(\cdot, s)\|_2^4 \|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})(\cdot, s)\|_2 \\
&\times \|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})(\cdot, s)\|_2\, ds.
\end{aligned}$$

Definamos

$$\mathcal{I}(t) := \sup_{T^*-\tau\le s\le t} \|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})(\cdot, s)\|_2 + \Big(\int_{T^*-\tau}^{t} \|(\nabla\nabla_h \mathbf{u}, \nabla\nabla_h \mathbf{w}, \nabla\nabla_h \mathbf{b})(\cdot, s)\|_2^2\, ds\Big)^{\frac{1}{2}}$$

e também

$$\mathcal{J}(t) := \sup_{T^*-\tau\le s\le t} \|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})(\cdot, s)\|_2 + \Big(\int_{T^*-\tau}^{t} \|(\Delta \mathbf{u}, \Delta \mathbf{w}, \Delta \mathbf{b})(\cdot, s)\|_2^2\, ds\Big)^{\frac{1}{2}}.$$

Assim sendo, concluímos que

$$\begin{aligned}
\mathcal{I}^2(t) &\le 2\sup_{T^*-\tau\le s\le t} \|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})(\cdot, s)\|_2^2 + 2\int_{T^*-\tau}^{t} \|(\nabla\nabla_h \mathbf{u}, \nabla\nabla_h \mathbf{w}, \nabla\nabla_h \mathbf{b})(\cdot, s)\|_2^2\, ds \\
&\le C + C\int_{T^*-\tau}^{t} \|(\nabla u_3, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})(\cdot, s)\|_2^4 \|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})(\cdot, s)\|_2 \|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})(\cdot, s)\|_2\, ds \\
&\le C + C\mathcal{J}^{\frac{3}{4}}(t)\mathcal{I}(t)\int_{T^*-\tau}^{t} \|(\nabla u_3, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})(\cdot, s)\|_2^4 \|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})(\cdot, s)\|_2^{\frac{1}{4}}\, ds \\
&\le C + C\mathcal{J}^{\frac{3}{4}}(t)\mathcal{I}(t)\int_{0}^{T} [\|(\nabla u_3, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})(\cdot, s)\|_2^{\frac{32}{7}} + \|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})(\cdot, s)\|_2^2]\, ds,
\end{aligned}$$

onde nesta última desigualdade foi usada a Desigualdade de Young, com $p = \frac{8}{7}$ e $q = 8$. Note que, pela hipótese (3.3), (1.1) encontra-se

$$\mathcal{I}^2(t) \le C + C\mathcal{J}^{\frac{3}{2}}(t),$$

basta aplicar a Desigualdade de Young novamente. Logo,

$$\mathcal{I}(t) \le C + C\mathcal{J}^{\frac{3}{4}}(t). \tag{3.16}$$

Agora, estamos em condições de estimar $\|(\nabla\mathbf{u}, \nabla\mathbf{w}, \nabla\mathbf{b})\|_2$. Inicialmente, vejamos que

$$\frac{1}{2}\frac{d}{dt}\|(\nabla\mathbf{u}, \nabla\mathbf{w}, \nabla\mathbf{b})\|_2^2 = \frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|_2^2 + \frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{w}\|_2^2 + \frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{b}\|_2^2.$$

Desta forma, para alcançar o nosso objetivo, iremos estudar cada parcela do lado direito da igualdade acima. Observe que, pela primeira equação do sistema (1), temos

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|_2^2 &= (\nabla\partial_t\mathbf{u}, \nabla\mathbf{u})_2 \\
&= -(\partial_t\mathbf{u}, \Delta\mathbf{u})_2 \\
&= -(\mu+\chi)(\Delta\mathbf{u}, \Delta\mathbf{u})_2 + (\mathbf{u}\cdot\nabla\mathbf{u}, \Delta\mathbf{u})_2 - (\mathbf{b}\cdot\nabla\mathbf{b}, \Delta\mathbf{u})_2 \\
&\quad + (\nabla(p + \frac{1}{2}|\mathbf{b}|^2), \Delta\mathbf{u})_2 - \chi(\nabla\times\mathbf{w}, \Delta\mathbf{u})_2.
\end{aligned}$$

É fácil ver novamente que o termo que envolve a pressão é nulo. De fato, podemos escrever

$$\begin{aligned}
(\nabla(p + \frac{1}{2}|\mathbf{b}|^2, \Delta\mathbf{u}))_2 &= \sum_{i=1}^{3}(\partial_i(p + \frac{1}{2}|\mathbf{b}|^2, \Delta u_i)_2 \\
&= -\sum_{i=1}^{3}((p + \frac{1}{2}|\mathbf{b}|^2, \Delta\partial_i u_i)_2 \\
&= 0,
\end{aligned}$$

pois $\nabla\cdot\mathbf{u} = 0$. Assim,

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|_2^2 + (\mu+\chi)\|\Delta\mathbf{u}\|_2^2 = (\mathbf{u}\cdot\nabla\mathbf{u}, \Delta\mathbf{u})_2 - (\mathbf{b}\cdot\nabla\mathbf{b}, \Delta\mathbf{u})_2 - \chi(\nabla\times\mathbf{w}, \Delta\mathbf{u})_2. \tag{3.17}$$

Agora, note que pela segunda equação do sistema (1), obtemos

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{w}\|_2^2 &= (\nabla\partial_t\mathbf{w}, \nabla\mathbf{w})_2 \\
&= -(\partial_t\mathbf{w}, \Delta\mathbf{w})_2 \\
&= -\gamma(\Delta\mathbf{w}, \Delta\mathbf{w})_2 - \kappa(\nabla(\nabla\cdot\mathbf{w}), \Delta\mathbf{w})_2 + 2\chi(\mathbf{w}, \Delta\mathbf{w})_2 + (\mathbf{u}\cdot\nabla\mathbf{w}, \Delta\mathbf{w})_2 \\
&\quad -\chi(\nabla\times\mathbf{u}, \Delta\mathbf{w})_2.
\end{aligned}$$

Veja que, analogamente a (3.7), temos

$$\begin{aligned}
-(\nabla(\nabla\cdot\mathbf{w}),\Delta\mathbf{w})_2 &= -\sum_{i=1}^{3}(\partial_i(\nabla\cdot\mathbf{w}),\Delta w_i)_2 = \sum_{i=1}^{3}(\nabla\cdot\mathbf{w},\Delta(\partial_i w_i))_2\\
&= (\nabla\cdot\mathbf{w},\Delta(\nabla\cdot\mathbf{w}))_2 = \sum_{j=1}^{3}(\nabla\cdot\mathbf{w},\partial_j^2(\nabla\cdot\mathbf{w}))_2\\
&= -\sum_{j=1}^{3}(\partial_j(\nabla\cdot\mathbf{w}),\partial_j(\nabla\cdot\mathbf{w}))_2 = -(\nabla(\nabla\cdot\mathbf{w}),\nabla(\nabla\cdot\mathbf{w}))_2\\
&= -\|\nabla(\nabla\cdot\mathbf{w})\|_2^2
\end{aligned}$$

e, por integração por partes, obtemos

$$(\mathbf{w},\Delta\mathbf{w})_2 = \sum_{i=1}^{3}(\mathbf{w},\partial_i^2\mathbf{w}) = -\sum_{i=1}^{3}(\partial_i\mathbf{w},\partial_i\mathbf{w}) = -(\nabla\mathbf{w},\nabla\mathbf{w}) = -\|\nabla\mathbf{w}\|_2^2.$$

Assim, chegamos a seguinte igualdade:

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{w}\|_2^2 + \gamma\|\Delta\mathbf{w}\|_2^2 + \kappa\|\nabla(\nabla\cdot\mathbf{w})\|_2^2 + 2\chi\|\nabla\mathbf{w}\|_2^2 = (\mathbf{u}\cdot\nabla\mathbf{w},\Delta\mathbf{w})_2 - \chi(\nabla\times\mathbf{u},\Delta\mathbf{w})_2. \tag{3.18}$$

Por fim, veja que pela terceira equação do sistema (1), obtemos

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{b}\|_2^2 &= (\nabla\partial_t\mathbf{b},\nabla\mathbf{b})_2\\
&= -(\partial_t\mathbf{b},\Delta\mathbf{b})_2\\
&= -\nu(\Delta\mathbf{b},\Delta\mathbf{b})_2 + (\mathbf{u}\cdot\nabla\mathbf{b},\Delta\mathbf{b})_2 - (\mathbf{b}\cdot\nabla\mathbf{u},\Delta\mathbf{b})_2,
\end{aligned}$$

ou seja,

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{b}\|_2^2 + \nu\|\Delta\mathbf{b}\|_2^2 = (\mathbf{u}\cdot\nabla\mathbf{b},\Delta\mathbf{b})_2 - (\mathbf{b}\cdot\nabla\mathbf{u},\Delta\mathbf{b})_2. \tag{3.19}$$

Logo, somando (3.17), (3.18) e (3.19), obtemos

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^2 + (\mu+\chi)\|\Delta\mathbf{u}\|_2^2 + \gamma\|\Delta\mathbf{w}\|_2^2 + \nu\|\Delta\mathbf{b}\|_2^2 + \kappa\|\nabla(\nabla\cdot\mathbf{w})\|_2^2 + 2\chi\|\nabla\mathbf{w}\|_2^2\\
&= (\mathbf{u}\cdot\nabla\mathbf{u},\Delta\mathbf{u})_2 - (\mathbf{b}\cdot\nabla\mathbf{b},\Delta\mathbf{u})_2 - \chi(\nabla\times\mathbf{w},\Delta\mathbf{u})_2 + (\mathbf{u}\cdot\nabla\mathbf{w},\Delta\mathbf{w})_2 - \chi(\nabla\times\mathbf{u},\Delta\mathbf{w})_2\\
&+ (\mathbf{u}\cdot\nabla\mathbf{b},\Delta\mathbf{b})_2 - (\mathbf{b}\cdot\nabla\mathbf{u},\Delta\mathbf{b})_2.
\end{aligned}$$

A partir de agora, iremos estudar as parcelas do lado direito da desigualdade acima. Observe que,

$$
\begin{aligned}
((\mathbf{u}\cdot\nabla)\mathbf{u},\Delta\mathbf{u})_2 &= \sum_{i=1}^{3}\sum_{j=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_i u_j)\Delta u_j\,dx\\
&= \sum_{j=1}^{3}\int_{\mathbb{R}^3} u_3(\partial_3 u_j)\Delta_h u_j\,dx + \sum_{i=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_i u_j)\Delta u_j\,dx\\
&\quad + \sum_{j=1}^{3}\int_{\mathbb{R}^3} u_3(\partial_3 u_j)(\partial_3^2 u_j)\,dx\\
&=: K_1 + K_2 + K_3.
\end{aligned}
$$

A fim de estimar o produto interno do lado esquerdo da desigualdade acima, iremos explorar K_j para cada $j = 1, 2, 3$. Veja que, por integração por partes, temos

$$
\begin{aligned}
K_1 &:= \sum_{j=1}^{3}\int_{\mathbb{R}^3} u_3(\partial_3 u_j)\Delta_h u_j\,dx\\
&= \sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} u_3(\partial_3 u_j)(\partial_k^2 u_j)\,dx\\
&= -\sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} (\partial_k u_3)(\partial_3 u_j)(\partial_k u_j)\,dx - \sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} u_3(\partial_k\partial_3 u_j)(\partial_k u_j)\,dx.
\end{aligned}
$$

Porém, também por integração por partes, temos

$$
\begin{aligned}
-\sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} u_3(\partial_k\partial_3 u_j)(\partial_k u_j)\,dx &= \sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} (\partial_3 u_3)(\partial_k u_j)^2\,dx\\
&\quad + \sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} u_3(\partial_3\partial_k u_j)(\partial_k u_j)\,dx.
\end{aligned}
$$

Assim, podemos escrever

$$
-\sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} u_3(\partial_k\partial_3 u_j)(\partial_k u_j)\,dx = \frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} (\partial_3 u_3)(\partial_k u_j)^2\,dx. \tag{3.20}
$$

Logo, encontramos a seguinte expressão para K_1:

$$
K_1 = -\sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} (\partial_k u_3)(\partial_3 u_j)(\partial_k u_j)\,dx + \frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} (\partial_3 u_3)(\partial_k u_j)^2\,dx. \tag{3.21}
$$

Agora, vejamos que

$$
\begin{aligned}
K_2 &:= \sum_{i=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_i u_j)\Delta u_j\,dx \\
&= \sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_i u_j)(\partial_k^2 u_j)\,dx \\
&= -\sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3} (\partial_k u_i)(\partial_i u_j)(\partial_k u_j)\,dx - \sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_k\partial_i u_j)(\partial_k u_j)\,dx.
\end{aligned}
$$

De forma análoga a (3.20), obtemos

$$
-\sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_k\partial_i u_j)(\partial_k u_j)\,dx = \frac{1}{2}\sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3} (\partial_i u_i)(\partial_k u_j)^2\,dx.
$$

Assim,

$$
K_2 = -\sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3} (\partial_k u_i)(\partial_i u_j)(\partial_k u_j)\,dx + \frac{1}{2}\sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3} (\partial_i u_i)(\partial_k u_j)^2\,dx. \tag{3.22}
$$

E, por fim, observe que, analogamente a (3.20), chegamos a

$$
K_3 = \sum_{j=1}^{3}\int_{\mathbb{R}^3} u_3(\partial_3 u_j)(\partial_3^2 u_j)\,dx = -\frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3} (\partial_3 u_3)(\partial_3 u_j)^2\,dx.
$$

Como $\nabla\cdot\mathbf{u} = 0$, ou seja, $\partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 = 0$, então $\partial_3 u_3 = -\partial_1 u_1 - \partial_2 u_2$. Com isso,

$$
K_3 = \frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3} (\partial_1 u_1 + \partial_2 u_2)(\partial_3 u_j)^2\,dx. \tag{3.23}
$$

Logo, combinando (3.21)-(3.23), obtemos

$$
\begin{aligned}
&\int_{\mathbb{R}^3} (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\Delta\mathbf{u}\,dx \le \sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} |\partial_k u_3||\partial_3 u_j||\partial_k u_j|\,dx + \frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} |\partial_3 u_3||\partial_k u_j|^2\,dx \\
&+ \sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3} |\partial_k u_i||\partial_i u_j||\partial_k u_j|\,dx + \frac{1}{2}\sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3} |\partial_i u_i||\partial_k u_j|^2\,dx \\
&+ \frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3} |\partial_1 u_1||\partial_3 u_j|^2\,dx + \frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3} |\partial_2 u_2||\partial_3 u_j|^2\,dx.
\end{aligned}
$$

Portanto,

$$\int_{\mathbb{R}^3} (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\Delta\mathbf{u}\,dx \leq C\int_{\mathbb{R}^3} |\nabla_h\mathbf{u}||\nabla\mathbf{u}|^2\,dx.$$

Agora, vejamos que

$$\begin{aligned}
((\mathbf{u}\cdot\nabla)\mathbf{w},\Delta\mathbf{w})_2 &= \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_i w_j)(\partial_k^2 w_j)\,dx\\
&= \sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} u_3(\partial_3 w_j)(\partial_k^2 w_j)\,dx + \sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3} u_i(\partial_i w_j)(\partial_k^2 w_j)\,dx\\
&\quad+\sum_{j=1}^{3}\int_{\mathbb{R}^3} u_3(\partial_3 w_j)(\partial_3^2 w_j)\,dx\\
&=: R_1 + R_2 + R_3.
\end{aligned}$$

Assim, com o intuito de estimar o produto interno acima, iremos estudar R_j com $j \in \{1,2,3\}$. Vejamos que, por integração por partes, temos

$$\begin{aligned}
R_1 &:= \sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} u_3(\partial_3 w_j)(\partial_k^2 w_j)\,dx\\
&= -\sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} (\partial_k u_3)(\partial_3 w_j)(\partial_k w_j)\,dx - \sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} u_3(\partial_k\partial_3 w_j)(\partial_k w_j)\,dx.
\end{aligned}$$

Mas, é fácil notar que

$$\begin{aligned}
-\sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} u_3(\partial_k\partial_3 w_j)(\partial_k w_j)\,dx &= \sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} (\partial_3 u_3)(\partial_k w_j)^2\,dx\\
&\quad+\sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} u_3(\partial_3\partial_k w_j)(\partial_k w_j)\,dx.
\end{aligned}$$

Desta forma,

$$-\sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} u_3(\partial_k\partial_3 w_j)(\partial_k w_j)\,dx = \frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} (\partial_3 u_3)(\partial_k w_j)^2\,dx.$$

Logo,

$$R_1 = -\sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}(\partial_k u_3)(\partial_3 w_j)(\partial_k w_j)\,dx + \frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}(\partial_3 u_3)(\partial_k w_j)^2\,dx$$
$$\leq C\int_{\mathbb{R}^3}|\nabla_h u_3||\nabla \mathbf{w}||\nabla_h \mathbf{w}|\,dx + C\int_{\mathbb{R}^3}|\nabla u_3||\nabla_h \mathbf{w}|^2\,dx. \tag{3.24}$$

Agora, note que

$$R_2 := \sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}u_i(\partial_i w_j)(\partial_k^2 w_j)\,dx$$
$$= -\sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}(\partial_k u_i)(\partial_i w_j)(\partial_k w_j)\,dx - \sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}u_i(\partial_k\partial_i w_j)(\partial_k w_j)\,dx.$$

Mas, por outro lado, temos que

$$-\sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}u_i(\partial_k\partial_i w_j)(\partial_k w_j)\,dx = \sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}(\partial_i u_i)(\partial_k w_j)^2\,dx$$
$$+\sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}u_i(\partial_i\partial_k w_j)(\partial_k w_j)\,dx.$$

Com isso,

$$-\sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}u_i(\partial_k\partial_i w_j)(\partial_k w_j)\,dx = \frac{1}{2}\sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}(\partial_i u_i)(\partial_k w_j)^2\,dx.$$

Logo,

$$R_2 = -\sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}(\partial_k u_i)(\partial_i w_j)(\partial_k w_j)\,dx + \frac{1}{2}\sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}(\partial_i u_i)(\partial_k w_j)^2\,dx$$
$$\leq C\int_{\mathbb{R}^3}|\nabla \mathbf{u}||\nabla_h \mathbf{w}||\nabla \mathbf{w}|\,dx + C\int_{\mathbb{R}^3}|\nabla_h \mathbf{u}||\nabla \mathbf{w}|^2\,dx. \tag{3.25}$$

E, por fim, note que

$$R_3 := \sum_{j=1}^{3}\int_{\mathbb{R}^3}u_3(\partial_3 w_j)(\partial_3^2 w_j)\,dx$$
$$= -\sum_{j=1}^{3}\int_{\mathbb{R}^3}(\partial_3 u_3)(\partial_3 w_j)^2\,dx - \sum_{j=1}^{3}\int_{\mathbb{R}^3}u_3(\partial_3 w_j)(\partial_3^2 w_j)\,dx.$$

Como $\nabla \cdot \mathbf{u} = 0$, ou seja, $\partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 = 0$, então $\partial_3 u_3 = -\partial_1 u_1 - \partial_2 u_2$. Desta forma,

$$R_3 = -\frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3} (\partial_3 u_3)(\partial_3 w_j)^2\,dx = \frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} (\partial_k u_k)(\partial_3 w_j)^2\,dx \leq C\int_{\mathbb{R}^3} |\nabla_h \mathbf{u}||\nabla \mathbf{w}|^2\,dx. \tag{3.26}$$

Assim, juntando (3.24)-(3.26), obtemos

$$((\mathbf{u}\cdot\nabla)\mathbf{w}, \Delta\mathbf{w})_2 \leq C\int_{\mathbb{R}^3} |\nabla\mathbf{u}||\nabla\mathbf{w}||\nabla_h\mathbf{w}|\,dx + C\int_{\mathbb{R}^3} |\nabla\mathbf{u}||\nabla_h\mathbf{w}|^2\,dx + C\int_{\mathbb{R}^3} |\nabla_h\mathbf{u}||\nabla\mathbf{w}|^2\,dx.$$

Analogamente, podemos encontrar

$$((\mathbf{u}\cdot\nabla)\mathbf{b}, \Delta\mathbf{b})_2 \leq C\int_{\mathbb{R}^3} |\nabla\mathbf{u}||\nabla\mathbf{b}||\nabla_h\mathbf{b}|\,dx + C\int_{\mathbb{R}^3} |\nabla\mathbf{u}||\nabla_h\mathbf{b}|^2\,dx + C\int_{\mathbb{R}^3} |\nabla_h\mathbf{u}||\nabla\mathbf{b}|^2\,dx.$$

Observe que,

$$\begin{aligned}
-(\mathbf{b}\cdot\nabla\mathbf{b}, \Delta\mathbf{u})_2 - (\mathbf{b}\cdot\nabla\mathbf{u}, \Delta\mathbf{b})_2 &= -\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} b_i(\partial_i b_j)(\partial_k^2 u_j)\,dx - \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} b_i(\partial_i u_j)(\partial_k^2 b_j)\,dx \\
&= \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} (\partial_k b_i)(\partial_i b_j)(\partial_k u_j)\,dx + \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} b_i(\partial_k\partial_i b_j)(\partial_k u_j)\,dx \\
&\quad + \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} (\partial_k b_i)(\partial_i u_j)(\partial_k b_j)\,dx + \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} b_i(\partial_k\partial_i u_j)(\partial_k b_j)\,dx.
\end{aligned}$$

Portanto, usando o fato que $\mathbf{b}$ é livre de divergente, encontramos

$$\begin{aligned}
-(\mathbf{b}\cdot\nabla\mathbf{b}, \Delta\mathbf{u})_2 - (\mathbf{b}\cdot\nabla\mathbf{u}, \Delta\mathbf{b})_2 &= \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} (\partial_k b_i)(\partial_i b_j)(\partial_k u_j)\,dx - \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} b_i(\partial_i\partial_k u_j)(\partial_k b_j)\,dx \\
&\quad + \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} (\partial_k b_i)(\partial_i u_j)(\partial_k b_j)\,dx + \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} b_i(\partial_k\partial_i u_j)(\partial_k b_j)\,dx.
\end{aligned}$$

Consequentemente,

$$
\begin{aligned}
-(\mathbf{b}\cdot\nabla\mathbf{b},\Delta\mathbf{u})_2-(\mathbf{b}\cdot\nabla\mathbf{u},\Delta\mathbf{b})_2 &= \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}(\partial_k b_i)(\partial_i b_j)(\partial_k u_j)\,dx+\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}(\partial_k b_i)(\partial_i u_j)(\partial_k b_j)\,dx\\
&=\sum_{j=1}^{3}\sum_{k=1}^{2}\left[\int_{\mathbb{R}^3}(\partial_k b_3)(\partial_3 b_j)(\partial_k u_j)\,dx+\int_{\mathbb{R}^3}(\partial_k b_3)(\partial_3 u_j)(\partial_k b_j)\,dx\right]\\
&\quad+\sum_{i=1}^{2}\sum_{j,k=1}^{3}\left[\int_{\mathbb{R}^3}(\partial_k b_i)(\partial_i b_j)(\partial_k u_j)\,dx+\int_{\mathbb{R}^3}(\partial_k b_i)(\partial_i u_j)(\partial_k b_j)\,dx\right]\\
&\quad+\sum_{j=1}^{3}\left[\int_{\mathbb{R}^3}(\partial_3 b_3)(\partial_3 b_j)(\partial_3 u_j)\,dx+\int_{\mathbb{R}^3}(\partial_3 b_3)(\partial_3 u_j)(\partial_3 b_j)\,dx\right]\\
&=: P_1+P_2+P_3.
\end{aligned}
$$

Com o intuito de conseguir uma estimativa para a diferença apresentada no lado esquerdo da igualdade acima, iremos analisar P_j para $j\in\{1,2,3\}$. Vejamos que, por integração por partes, obtemos

$$
\begin{aligned}
P_1 &:= \sum_{j=1}^{3}\sum_{k=1}^{2}\left[\int_{\mathbb{R}^3}(\partial_k b_3)(\partial_3 b_j)(\partial_k u_j)\,dx+\int_{\mathbb{R}^3}(\partial_k b_3)(\partial_3 u_j)(\partial_k b_j)\,dx\right]\\
&\leq C\int_{\mathbb{R}^3}|\nabla_h\mathbf{b}||\nabla\mathbf{b}||\nabla_h\mathbf{u}|\,dx+C\int_{\mathbb{R}^3}|\nabla_h\mathbf{b}|\;|\nabla\mathbf{u}||\nabla_h\mathbf{b}|\,dx. \qquad (3.27)
\end{aligned}
$$

Além disso,

$$
\begin{aligned}
P_2 &:= \sum_{i=1}^{2}\sum_{j,k=1}^{3}\left[\int_{\mathbb{R}^3}(\partial_k b_i)(\partial_i b_j)(\partial_k u_j)\,dx+\int_{\mathbb{R}^3}(\partial_k b_i)(\partial_i u_j)(\partial_k b_j)\,dx\right]\\
&\leq C\int_{\mathbb{R}^3}|\nabla\mathbf{b}||\nabla_h\mathbf{b}||\nabla\mathbf{u}|\,dx+C\int_{\mathbb{R}^3}|\nabla\mathbf{b}||\nabla_h\mathbf{u}||\nabla\mathbf{b}|\,dx \qquad (3.28)
\end{aligned}
$$

e também, pelo fato que $\nabla\cdot\mathbf{b}=0$, chegamos a

$$
\begin{aligned}
P_3 &:= \sum_{j=1}^{3}\left[\int_{\mathbb{R}^3}(\partial_3 b_3)(\partial_3 b_j)(\partial_3 u_j)\,dx+\int_{\mathbb{R}^3}(\partial_3 b_3)(\partial_3 u_j)(\partial_3 b_j)\,dx\right]\\
&= -\sum_{j=1}^{3}\sum_{k=1}^{2}\left[\int_{\mathbb{R}^3}(\partial_k b_k)(\partial_3 b_j)(\partial_3 u_j)\,dx+\int_{\mathbb{R}^3}(\partial_k b_k)(\partial_3 u_j)(\partial_3 b_j)\,dx\right]\\
&\leq C\int_{\mathbb{R}^3}|\nabla_h\mathbf{b}||\nabla\mathbf{u}||\nabla\mathbf{b}|\,dx. \qquad (3.29)
\end{aligned}
$$

Logo, combinando (3.27)-(3.29), temos

$$-(\mathbf{b}\cdot\nabla\mathbf{b},\Delta\mathbf{u})-(\mathbf{b}\cdot\nabla\mathbf{u},\Delta\mathbf{b})\leq C\int_{\mathbb{R}^3}|\nabla_h\mathbf{b}||\nabla\mathbf{b}||\nabla_h\mathbf{u}|\,dx+C\int_{\mathbb{R}^3}|\nabla_h\mathbf{b}||\nabla\mathbf{u}||\nabla_h\mathbf{b}|\,dx$$
$$+C\int_{\mathbb{R}^3}|\nabla\mathbf{b}||\nabla_h\mathbf{b}||\nabla\mathbf{u}|\,dx+C\int_{\mathbb{R}^3}|\nabla\mathbf{b}||\nabla_h\mathbf{u}||\nabla\mathbf{b}|\,dx.$$

Vimos em (2.54) que

$$\begin{aligned}-\chi(\nabla\times\mathbf{w},\Delta\mathbf{u})_2-\chi(\nabla\times\mathbf{u},\Delta\mathbf{w})_2 &\leq -2\chi(\nabla\times\mathbf{u},\Delta\mathbf{w})_2=2\chi(\nabla\times(\nabla\mathbf{u}),\nabla\mathbf{w})_2\\ &\leq \chi\|\nabla\mathbf{w}\|_2^2+\chi\|\Delta\mathbf{u}\|_2^2.\end{aligned}$$

Desta forma, pelo Lema 3.1, temos que

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^2+\mu\|\Delta\mathbf{u}\|_2^2+\gamma\|\Delta\mathbf{w}\|_2^2+\nu\|\Delta\mathbf{b}\|_2^2+\kappa\|\nabla(\nabla\cdot\mathbf{w})\|_2^2+\chi\|\nabla\mathbf{w}\|_2^2\\ &\leq C\int_{\mathbb{R}^3}|(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})||(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})|^2\,dx\\ &\leq C\|(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})\|_2\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^{\frac{1}{2}}\|(\nabla\nabla_h\mathbf{u},\nabla\nabla_h\mathbf{w},\nabla\nabla_h\mathbf{b})\|_2\|(\Delta\mathbf{u},\Delta\mathbf{w},\Delta\mathbf{b})\|_2^{\frac{1}{2}}.\end{aligned}$$

Por conseguinte, podemos escrever

$$\begin{aligned}&\frac{d}{dt}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^2+2\alpha\|(\Delta\mathbf{u},\Delta\mathbf{w},\Delta\mathbf{b})\|_2^2+2\kappa\|\nabla(\nabla\cdot\mathbf{w})\|_2^2+2\chi\|\nabla\mathbf{w}\|_2^2\\ &\leq C\|(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})\|_2\;\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^{\frac{1}{2}}\;\|(\nabla\nabla_h\mathbf{u},\nabla\nabla_h\mathbf{w},\nabla\nabla_h\mathbf{b})\|_2\|(\Delta\mathbf{u},\Delta\mathbf{w},\Delta\mathbf{b})\|_2^{\frac{1}{2}},\end{aligned}$$

onde $\alpha=\min\{\mu,\gamma,\nu\}$. Integrando a desigualdade acima sobre $[T^*-\tau,s]$, onde $s\leq t$, e utilizando a desigualdade de Hölder obtemos

$$\begin{aligned}&\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\cdot,s)\|_2^2+2\alpha\int_{T^*-\tau}^{s}\|(\Delta\mathbf{u},\Delta\mathbf{w},\Delta\mathbf{b})(\cdot,\tau)\|_2^2\,d\tau+2\kappa\int_{T^*-\tau}^{s}\|\nabla(\nabla\cdot\mathbf{w})(\cdot,\tau)\|_2^2\,d\tau\\ &+2\chi\int_{T^*-\tau}^{s}\|\nabla\mathbf{w}(\cdot,\tau)\|_2^2\,d\tau\leq C+C\mathcal{I}(t)\left(\int_{T^*-\tau}^{s}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\cdot,\tau)\|_2^2\,d\tau\right)^{\frac{1}{4}}\\ &\times\left(\int_{T^*-\tau}^{s}\|(\nabla\nabla_h\mathbf{u},\nabla\nabla_h\mathbf{w},\nabla\nabla_h\mathbf{b})(\cdot,\tau)\|_2^2\right)^{\frac{1}{2}}\left(\int_{T^*-\tau}^{s}\|(\Delta\mathbf{u},\Delta\mathbf{w},\Delta\mathbf{b})(\cdot,\tau)\|_2^2\,d\tau\right)^{\frac{1}{4}}\\ &\leq C+C\mathcal{J}^{\frac{1}{2}}(t)\mathcal{I}^2(t)\left(\int_{T^*-\tau}^{s}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\cdot,\tau)\|_2^2\,d\tau\right)^{\frac{1}{4}}.\end{aligned}$$

Utilizando a Desigualdade de Young, chegamos a

$$
\begin{aligned}
\mathcal{J}^2(t) &\leq C + C\mathcal{I}^2(t)\mathcal{J}^{\frac{1}{2}}(t)\left(\int_{T^*-\tau}^{t} \|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})(\cdot,\tau)\|_2^2 \, d\tau\right)^{\frac{1}{4}} \\
&\leq C + \mathcal{I}^{\frac{8}{3}}(t)\left(\int_{T^*-\tau}^{t} \|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})(\cdot,\tau)\|_2^2 \, d\tau\right)^{\frac{1}{3}} + \frac{1}{2}\mathcal{J}^2(t).
\end{aligned}
$$

Assim, infere-se, por (3.16), que

$$
\begin{aligned}
\mathcal{J}(t) &\leq C + \mathcal{I}^{\frac{4}{3}}(t)\left(\int_{T^*-\tau}^{t} \|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})(\cdot,\tau)\|_2^2 \, d\tau\right)^{\frac{1}{6}} \\
&\leq C + C(C + C\mathcal{J}^{\frac{3}{4}}(t))^{\frac{4}{3}}\left(\int_{T^*-\tau}^{t} \|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})(\cdot,\tau)\|_2^2 \, d\tau\right)^{\frac{1}{6}} \\
&\leq C + (C + C\mathcal{J}(t))\left(\int_{T^*-\tau}^{t} \|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})(\cdot,\tau)\|_2^2 \, d\tau\right)^{\frac{1}{6}}.
\end{aligned}
$$

Por utilizar (1.1) e considerar $0 < \tau \ll 1$ tal que

$$
\left(\int_{T^*-\tau}^{T^*} \|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})(\cdot,\tau)\|_2^2 \, d\tau\right)^{\frac{1}{6}} < \frac{1}{2C},
$$

obtemos

$$
\mathcal{J}(t) \leq C + (C + C\mathcal{J}(t))\frac{1}{2C} \leq C + \frac{1}{2}\mathcal{J}(t), \quad \forall\, T^* - \tau \leq t < T^*.
$$

Com isso, chegamos a

$$
\mathcal{J}(t) \leq C.
$$

Logo,

$$
\|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})(\cdot,t)\|_2 \leq C, \quad \forall\, T^* - \tau \leq t < T^*.
$$

Isto é um absurdo (ver desigualdade de Leray em [3]). E isto conclui a prova do teorema. □

3.2 Critério de Regularidade Envolvendo $(\partial_3 u_3, \partial_3 \mathbf{w}, \partial_3 \mathbf{b})(\cdot,t)$

Nesta seção, nosso intuito é exibir mais um critério de regularidade para uma solução fraca $(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot,t)$ do sistema de equações magneto-micropolar (1). Tal critério é uma extensão do resultado envolvendo uma solução fraca $\mathbf{u}(\cdot,t)$ das equações de Navier-Stokes (3), o qual garante a suavidade de $\mathbf{u}(\cdot,t)$ em $(0,T)$, quando adotamos a seguinte hipótese:

$$\partial_3 u_3(\cdot,t) \in L^\infty(0,T;L^2(\mathbb{R}^3)).$$

(Para mais detalhes ver [74]). Mais precisamente, provaremos a mesma implicação para $(\mathbf{u},\mathbf{w},\mathbf{b})(\cdot,t)$, quando assumirmos que

$$(\partial_3 u_3, \partial_3 \mathbf{w}, \partial_3 \mathbf{b})(\cdot,t) \in L^\infty(0,T;L^2(\mathbb{R}^3)).$$

Em ordem a obtermos êxito, provaremos a seguir um lema que desempenha papel importante na demonstração do resultado principal desta seção.

Lema 3.2 (ver [74])**.** *Sejam $f \in L^6(\mathbb{R}^3), g \in L^2(\mathbb{R}^3)$ tais que $\partial_3 f \in L^2(\mathbb{R}^3)$, $\nabla_h g \in L^2(\mathbb{R}^3)$. Então, vale a seguinte desigualdade:*

$$\int_{\mathbb{R}^3} [f(x)]^2 [g(x)]^2 \, dx \leq 2\sqrt{2} \|f\|_6^{\frac{3}{2}} \|\partial_3 f\|_2^{\frac{1}{2}} \|g\|_2 \|\nabla_h g\|_2. \tag{3.30}$$

Demonstração. Suponha, sem perda de generalidade, que $f,g \in C_c^\infty(\mathbb{R}^3)$. Primeiramente, vamos provar que a seguinte desigualdade é válida:

$$\|g(\cdot,\cdot,x_3)\|_4^4 \leq 2\|g(\cdot,\cdot,x_3)\|_2^2 \|\nabla_h g(\cdot,\cdot,x_3)\|_2^2. \tag{3.31}$$

Com efeito, observe que, pelo Teorema Fundamental do Cálculo, chegamos a

$$g^2(x) = 2\int_{-\infty}^{x_k} g(x_1,x_2,x_3)\partial_k g(x_1,x_2,x_3)\, dx_k, \quad k = 1,2.$$

Assim sendo, temos que

$$\begin{aligned} \max_{x_k \in \mathbb{R}} g^2(x_1,x_2,x_3) &= 2 \max_{x_k \in \mathbb{R}} \int_{-\infty}^{x_k} g(x_1,x_2,x_3)\partial_k g(x_1,x_2,x_3)\, dx_k \\ &\leq 2 \int_{\mathbb{R}} |g(x_1,x_2,x_3)\partial_k g(x_1,x_2,x_3)|\, dx_k, \end{aligned} \tag{3.32}$$

onde $k = 1, 2$. Desta forma, por (3.32) e pela Desigualdade de Hölder, obtemos

$$\begin{aligned}
\int_{\mathbb{R}^2} g^4(x_1,x_2,x_3)\,dx_1dx_2 &\le \int_{\mathbb{R}^2} [\max_{x_1\in\mathbb{R}} g^2(x_1,x_2,x_3)][\max_{x_2\in\mathbb{R}} g^2(x_1,x_2,x_3)]\,dx_1dx_2 \\
&= \Big(\int_{\mathbb{R}} \max_{x_2\in\mathbb{R}} g^2(x_1,x_2,x_3)\,dx_1\Big)\Big(\int_{\mathbb{R}} \max_{x_1\in\mathbb{R}} g^2(x_1,x_2,x_3)\,dx_2\Big) \\
&\le 4\int_{\mathbb{R}^2} |g(x_1,x_2,x_3)\partial_2 g(x_1,x_2,x_3)|\,dx_1dx_2 \\
&\quad \times \int_{\mathbb{R}^2} |g(x_1,x_2,x_3)\partial_1 g(x_1,x_2,x_3)|\,dx_1dx_2 \\
&\le 4\Big(\int_{\mathbb{R}^2} |g(x_1,x_2,x_3)|^2\,dx_1dx_2\Big)\Big(\int_{\mathbb{R}^2} |\partial_1 g(x_1,x_2,x_3)|^2\,dx_1dx_2\Big)^{\frac{1}{2}} \\
&\quad \times \Big(\int_{\mathbb{R}^2} |\partial_2 g(x_1,x_2,x_3)|^2\,dx_1dx_2\Big)^{\frac{1}{2}}.
\end{aligned}$$

Portanto, pela Desigualdade de Young, infere-se que

$$\begin{aligned}
\int_{\mathbb{R}^2} g^4(x_1,x_2,x_3)\,dx_1dx_2 &\le 4\Big(\int_{\mathbb{R}^2} |g(x_1,x_2,x_3)|^2\,dx_1dx_2\Big)\Big(\int_{\mathbb{R}^2} |\partial_1 g(x_1,x_2,x_3)|^2\,dx_1dx_2\Big)^{\frac{1}{2}} \\
&\quad \times \Big(\int_{\mathbb{R}^2} |\partial_2 g(x_1,x_2,x_3)|^2\,dx_1dx_2\Big)^{\frac{1}{2}} \\
&\le 2\Big(\int_{\mathbb{R}^2} |g(x_1,x_2,x_3)|^2\,dx_1dx_2\Big)\Big[\int_{\mathbb{R}^2} |\partial_1 g(x_1,x_2,x_3)|^2\,dx_1dx_2 \\
&\quad + \int_{\mathbb{R}^2} |\partial_2 g(x_1,x_2,x_3)|^2\,dx_1dx_2\Big] \\
&\le 2\Big(\int_{\mathbb{R}^2} |g(x_1,x_2,x_3)|^2\,dx_1dx_2\Big)\Big(\int_{\mathbb{R}^2} |\nabla_h g(x_1,x_2,x_3)|^2\,dx_1dx_2\Big).
\end{aligned}$$

E isto conclui a prova da desigualdade (3.31). Por usar a Desigualdades de Hölder e (3.31), encontramos

$$\begin{aligned}
\int_{\mathbb{R}^3} f^2g^2\,dx &\le \int_{\mathbb{R}} \|f(\cdot,\cdot,x_3)\|_4^2\|g(\cdot,\cdot,x_3)\|_4^2\,dx_3 \\
&\le \sqrt{2}\int_{\mathbb{R}} \|f(\cdot,\cdot,x_3)\|_4^2\|g(\cdot,\cdot,x_3)\|_2\|\nabla_h g(\cdot,\cdot,x_3)\|_2\,dx_3 \\
&\le \sqrt{2}\max_{x_3\in\mathbb{R}} \|f(\cdot,\cdot,x_3)\|_4^2\left(\int_{\mathbb{R}^3} \|g(\cdot,\cdot,x_3)\|_2^2\,dx_3\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3} \|\nabla_h g(\cdot,\cdot,x_3)\|_2^2\,dx_3\right)^{\frac{1}{2}}.
\end{aligned}$$

Dessa forma, chegamos a

$$\int_{\mathbb{R}^3} f^2g^2\,dx \le \sqrt{2}\max_{x_3\in\mathbb{R}} \|f(\cdot,\cdot,x_3)\|_4^2\|g\|_2\|\nabla_h g\|_2.$$

Agora, vamos estimar o máximo encontrado no lado direito da desigualdade acima. Primieramente, note que

$$4\int_{\mathbb{R}^2}\int_{-\infty}^{x_3}(f^3\partial_3 f)(x_1,x_2,s)\,ds=\int_{\mathbb{R}^2}\int_{-\infty}^{x_3}\partial_3(f^4)(x_1,x_2,s)\,dx_1dx_2ds=\|f(\cdot,\cdot,x_3)\|_4^4,$$

onde usamos o Teorema Fundamental do Cálculo e o fato de $f\in C_c^\infty(\mathbb{R}^3)$. Consequentemente, encontramos

$$\max_{x_3\in\mathbb{R}}\|f(\cdot,\cdot,x_3)\|_4^2\le 2\left(\int_{\mathbb{R}^3}|f|^3|\partial_3 f|\,dx\right)^{\frac{1}{2}}.$$

Logo, pela Desigualdade de Hölder, temos

$$\begin{aligned}\int_{\mathbb{R}^3}f^2g^2\,dx&\le 2\sqrt{2}\left(\int_{\mathbb{R}^2}|f|^3|\partial_3 f|\,dx\right)^{\frac{1}{2}}\|g\|_2\|\nabla_h g\|_2\\&\le 2\sqrt{2}\left(\int_{\mathbb{R}^2}|f|^6\,dx\right)^{\frac{1}{4}}\left(\int_{\mathbb{R}^3}|\partial_3 f|^2\,dx\right)^{\frac{1}{4}}\|g\|_2\|\nabla_h g\|_2\\&=2\sqrt{2}\|f\|_6^{\frac{3}{2}}\|\partial_3 f\|_2^{\frac{1}{2}}\|g\|_2\|\nabla_h g\|_2.\end{aligned}$$

Isto prova a desigualdade (3.30). □

Agora, permita-nos enunciar o resultado principal desta seção.

Teorema 3.2. *Sejam* $(\mathbf{u_0},\mathbf{w_0},\mathbf{b_0})\in L^2(\mathbb{R}^3)$ *com* $\nabla\cdot\mathbf{u_0}=\nabla\cdot\mathbf{b_0}=0$. *Assuma que* $T>0$. *Considere que* $(\mathbf{u},\mathbf{w},\mathbf{b})(\cdot,t)$ *é uma solução fraca das equações magneto-micropolares* (1) *em* $[0,T]$ *com condição inicial* $(\mathbf{u_0},\mathbf{w_0},\mathbf{b_0})$. *Se*

$$(\partial_3 u_3,\partial_3\mathbf{w},\partial_3\mathbf{b})(\cdot,t)\in L^\infty(0,T;L^2(\mathbb{R}^3)),\tag{3.33}$$

então $(\mathbf{u},\mathbf{w},\mathbf{b})$ *é suave em* $\mathbb{R}^3\times(0,T)$.

Demonstração. A mesma discussão estabelecida no início da demonstração do Teorema 3.1 será considerada aqui. Assim sendo, vimos na demonstração do Teorema 3.1, em (3.10), que

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})\|_2^2+(\mu+\chi)\|\nabla\nabla_h\mathbf{u}\|_2^2+\gamma\|\nabla\nabla_h\mathbf{w}\|_2^2+\nu\|\nabla\nabla_h\mathbf{b}\|_2^2+\kappa\|\nabla_h(\nabla\cdot\mathbf{w})\|_2^2\\&+2\chi\|\nabla_h\mathbf{w}\|_2^2=(\mathbf{u}\cdot\nabla\mathbf{u},\Delta_h\mathbf{u})_2-(\mathbf{b}\cdot\nabla\mathbf{b},\Delta_h\mathbf{u})_2-\chi(\nabla\times\mathbf{w},\Delta_h\mathbf{u})_2+(\mathbf{u}\cdot\nabla\mathbf{w},\Delta_h\mathbf{w})_2\\&-\chi(\nabla\times\mathbf{u},\Delta_h\mathbf{w})_2+(\mathbf{u}\cdot\nabla\mathbf{b},\Delta_h\mathbf{b})_2-(\mathbf{b}\cdot\nabla\mathbf{u},\Delta_h\mathbf{b})_2.\end{aligned}$$

Agora, iremos analisar cada parcela do lado direito da igualdade anterior. Afirmamos que o primeiro termo satisfaz a desigualdade a seguir:

$$(\mathbf{u}\cdot\nabla\mathbf{u},\Delta_h\mathbf{u})_2 \leq C\int_{\mathbb{R}^3}|u_3||\nabla\mathbf{u}||\nabla\nabla_h\mathbf{u}|\,dx.$$

Com efeito, já vimos na demonstração do Teorema 3.1, em (3.11), que

$$(\mathbf{u}\cdot\nabla\mathbf{u},\Delta_h\mathbf{u})_2 = \sum_{i,j=1}^{2}\int_{\mathbb{R}^3}u_i(\partial_i u_j)\Delta_h u_j\,dx + \sum_{j=1}^{2}\int_{\mathbb{R}^3}u_3(\partial_3 u_j)\Delta_h u_j\,dx + \sum_{i=1}^{3}\int_{\mathbb{R}^3}u_i(\partial_i u_3)\Delta_h u_3\,dx.$$

Assim, pela Afirmação 3.1 e por integração por partes, temos que

$$\begin{aligned}(\mathbf{u}\cdot\nabla\mathbf{u},\Delta_h\mathbf{u})_2 &= \frac{1}{2}\sum_{j,k=1}^{2}\int_{\mathbb{R}^3}(\partial_3 u_3)(\partial_k u_j)^2\,dx - \int_{\mathbb{R}^3}(\partial_1 u_1)(\partial_2 u_2)(\partial_3 u_3)\,dx + \int_{\mathbb{R}^3}(\partial_2 u_1)(\partial_1 u_2)(\partial_3 u_3)\,dx\\
&\quad + \sum_{j,k=1}^{2}\int_{\mathbb{R}^3}u_3(\partial_3 u_j)(\partial_k^2 u_j)\,dx + \sum_{i=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}u_i(\partial_i u_3)(\partial_k^2 u_3)\,dx\\
&= \frac{1}{2}\sum_{j,k=1}^{2}\int_{\mathbb{R}^3}(\partial_3 u_3)(\partial_k u_j)^2\,dx - \int_{\mathbb{R}^3}(\partial_1 u_1)(\partial_2 u_2)(\partial_3 u_3)\,dx + \int_{\mathbb{R}^3}(\partial_2 u_1)(\partial_1 u_2)(\partial_3 u_3)\,dx\\
&\quad + \sum_{j,k=1}^{2}\int_{\mathbb{R}^3}u_3(\partial_3 u_j)(\partial_k^2 u_j)\,dx - \sum_{i=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}(\partial_k u_i)(\partial_i u_3)(\partial_k u_3)\,dx\\
&\quad - \sum_{i=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}u_i(\partial_k\partial_i u_3)(\partial_k u_3)\,dx.\end{aligned}$$

Observe que, também por integração por partes, obtemos

$$\begin{aligned}-\sum_{i=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}u_i(\partial_k\partial_i u_3)(\partial_k u_3)\,dx &= \sum_{i=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}(\partial_i u_i)(\partial_k u_3)^2\,dx + \sum_{i=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}u_i(\partial_i\partial_k u_3)(\partial_k u_3)\,dx\\
&= \sum_{i=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}u_i(\partial_i\partial_k u_3)(\partial_k u_3)\,dx,\end{aligned}$$

pois $\mathbf{u}$ é livre de divergente. Consequentemente,

$$-\sum_{i=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}u_i(\partial_k\partial_i u_3)(\partial_k u_3)\,dx = 0.$$

Assim, por integração por partes, obtemos

$$
\begin{aligned}
(\mathbf{u}\cdot\nabla\mathbf{u},\Delta_h\mathbf{u})_2 = {} & \frac{1}{2}\sum_{j,k=1}^{2}\int_{\mathbb{R}^3}(\partial_3 u_3)(\partial_k u_j)^2\,dx - \int_{\mathbb{R}^3}(\partial_1 u_1)(\partial_2 u_2)(\partial_3 u_3)\,dx \\
& + \int_{\mathbb{R}^3}(\partial_2 u_1)(\partial_1 u_2)(\partial_3 u_3)\,dx + \sum_{j,k=1}^{2}\int_{\mathbb{R}^3}u_3(\partial_3 u_j)(\partial_k^2 u_j)\,dx \\
& - \sum_{i=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}(\partial_k u_i)(\partial_i u_3)(\partial_k u_3)\,dx,
\end{aligned}
$$

e consequentemente,

$$
\begin{aligned}
(\mathbf{u}\cdot\nabla\mathbf{u},\Delta_h\mathbf{u})_2 = {} & -\sum_{j,k=1}^{2}\int_{\mathbb{R}^3}u_3(\partial_3\partial_k u_j)(\partial_k u_j)\,dx + \int_{\mathbb{R}^3}(\partial_3\partial_1 u_1)(\partial_2 u_2)u_3\,dx \\
& + \int_{\mathbb{R}^3}(\partial_1 u_1)(\partial_3\partial_2 u_2)u_3\,dx - \int_{\mathbb{R}^3}(\partial_3\partial_2 u_1)(\partial_1 u_2)u_3\,dx \\
& - \int_{\mathbb{R}^3}(\partial_2 u_1)(\partial_3\partial_1 u_2)u_3\,dx + \sum_{j,k=1}^{2}\int_{\mathbb{R}^3}u_3(\partial_3 u_j)(\partial_k^2 u_j)\,dx \\
& + \sum_{i=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}(\partial_k^2 u_i)(\partial_i u_3)u_3\,dx + \sum_{i=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}(\partial_k u_i)(\partial_k\partial_i u_3)u_3\,dx.
\end{aligned}
$$

Deste modo, podemos escrever

$$
(\mathbf{u}\cdot\nabla\mathbf{u},\Delta_h\mathbf{u})_2 \le C\int_{\mathbb{R}^3}|u_3||\nabla\mathbf{u}||\nabla\nabla_h\mathbf{u}|\,dx.
$$

Observemos, agora, que

$$
\begin{aligned}
(\mathbf{u}\cdot\nabla\mathbf{w},\Delta_h\mathbf{w})_2 &= \sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}u_i(\partial_i w_j)(\partial_k^2 w_j)\,dx \\
&= -\sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}(\partial_k u_i)(\partial_i w_j)(\partial_k w_j)\,dx - \sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}u_i(\partial_k\partial_i w_j)(\partial_k w_j)\,dx.
\end{aligned}
$$

Mas, por integração por partes, obtemos

$$
\begin{aligned}
-\sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} u_i(\partial_k\partial_i w_j)(\partial_k w_j)\,dx = &\sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}(\partial_i u_i)(\partial_k w_j)^2\,dx \\
&+\sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} u_i(\partial_i\partial_k w_j)(\partial_k w_j)\,dx.
\end{aligned}
$$

Com isso, chegamos a

$$
-\sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} u_i(\partial_k\partial_i w_j)(\partial_k w_j)\;dx = 0,
$$

pois $\nabla\cdot\mathbf{u}=0$. Assim, podemos escrever

$$
\begin{aligned}
(\mathbf{u}\cdot\nabla\mathbf{w},\Delta_h\mathbf{w})_2 &= -\sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}(\partial_k u_i)(\partial_i w_j)\partial_k w_j\,dx \\
&= \sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}(\partial_i\partial_k u_i)(\partial_k w_j)w_j\,dx + \sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}(\partial_k u_i)(\partial_i\partial_k w_j)w_j\,dx \\
&= \sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}(\partial_k u_i)(\partial_i\partial_k w_j)w_j\,dx \\
&\le C\int_{\mathbb{R}^3}|\mathbf{w}||\nabla\mathbf{u}||\nabla\nabla_h\mathbf{w}|\,dx,
\end{aligned}
$$

já que $\mathbf{u}$ é livre de divergente. Analogamente, obtemos

$$
(\mathbf{u}\cdot\nabla\mathbf{b},\Delta_h\mathbf{b})_2 \le C\int_{\mathbb{R}^3}|\mathbf{b}||\nabla\mathbf{u}||\nabla\nabla_h\mathbf{b}|\,dx.
$$

Note também que

$$
\begin{aligned}
-(\mathbf{b}\cdot\nabla\mathbf{b},\Delta_h\mathbf{u})_2-(\mathbf{b}\cdot\nabla\mathbf{u},\Delta_h\mathbf{b})_2 &= -\sum_{i,j=1}^{3}\sum_{k=1}^{2}\left[\int_{\mathbb{R}^3} b_i(\partial_i b_j)(\partial_k^2 u_j)\,dx-\int_{\mathbb{R}^3} b_i(\partial_i u_j)(\partial_k^2 b_j)\,dx\right] \\
&\le C\int_{\mathbb{R}^3}|\mathbf{b}||\nabla\mathbf{b}||\nabla\nabla_h\mathbf{u}|\,dx + C\int_{\mathbb{R}^3}|\mathbf{b}||\nabla\mathbf{u}||\nabla\nabla_h\mathbf{b}|\,dx.
\end{aligned}
$$

Por fim, é verdade que

$$
-\chi(\nabla\times\mathbf{w},\Delta_h\mathbf{u})_2-\chi(\nabla\times\mathbf{u},\Delta_h\mathbf{w})_2 \le \chi\|\nabla\nabla_h\mathbf{u}\|_2^2+\chi\|\nabla_h\mathbf{w}\|_2^2,
$$

ver (3.15). Por conseguinte, chegamos a

$$
\begin{aligned}
&\frac{d}{dt}\|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})\|_2^2 + 2\alpha\|(\nabla\nabla_h \mathbf{u}, \nabla\nabla_h \mathbf{w}, \nabla\nabla_h \mathbf{b})\|_2^2 + 2\kappa\|\nabla_h(\nabla\cdot\mathbf{w})\|_2^2 + 2\chi\|\nabla_h \mathbf{w}\|_2^2 \\
&\leq C\int_{\mathbb{R}^3} |(u_3, \mathbf{w}, \mathbf{b})|\ |(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})|\ |(\nabla\nabla_h \mathbf{u}, \nabla\nabla_h \mathbf{w}, \nabla\nabla_h \mathbf{b})|\, dx, \qquad (3.34)
\end{aligned}
$$

onde $\alpha = \min\{\mu, \gamma, \nu\}$. Pela Desigualdade de Young, obtemos

$$
\begin{aligned}
&\frac{d}{dt}\|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})\|_2^2 + \alpha\|(\nabla\nabla_h \mathbf{u}, \nabla\nabla_h \mathbf{w}, \nabla\nabla_h \mathbf{b})\|_2^2 + 2\kappa\|\nabla_h(\nabla\cdot\mathbf{w})\|_2^2 + 2\chi\|\nabla_h \mathbf{w}\|_2^2 \\
&\leq C\int_{\mathbb{R}^3} |(u_3, \mathbf{w}, \mathbf{b})|^2\ |(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})|^2\ dx.
\end{aligned}
$$

Assim, pelo Lema 3.2 e Lema 2.1, e por usar a hipótese (3.33), encontramos

$$
\begin{aligned}
&\frac{d}{dt}\|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})\|_2^2 + \alpha\|(\nabla\nabla_h \mathbf{u}, \nabla\nabla_h \mathbf{w}, \nabla\nabla_h \mathbf{b})\|_2^2 + 2\kappa\|\nabla_h(\nabla\cdot\mathbf{w})\|_2^2 + 2\chi\|\nabla_h \mathbf{w}\|_2^2 \\
&\leq C\int_{\mathbb{R}^3} |(u_3, \mathbf{w}, \mathbf{b})|^2\ |(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})|^2\ dx \\
&\leq C\|(u_3, \mathbf{w}, \mathbf{b})\|_6^{\frac{3}{2}}\|(\partial_3 u_3, \partial_3 \mathbf{w}, \partial_3 \mathbf{b})\|_2^{\frac{1}{2}}\|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})\|_2\|(\nabla\nabla_h \mathbf{u}, \nabla\nabla_h \mathbf{w}, \nabla\nabla_h \mathbf{b})\|_2 \\
&\leq C\|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})\|_2\|(\partial_3 u_3, \partial_3 \mathbf{w}, \partial_3 \mathbf{b})\|_2\|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})\|_2\|(\nabla\nabla_h \mathbf{u}, \nabla\nabla_h \mathbf{w}, \nabla\nabla_h \mathbf{b})\|_2 \\
&\leq C\|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})\|_2\ \|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})\|_2\ \|(\nabla\nabla_h \mathbf{u}, \nabla\nabla_h \mathbf{w}, \nabla\nabla_h \mathbf{b})\|_2.
\end{aligned}
$$

Desta forma, por aplicar a Desigualdade de Young, obtemos

$$
\begin{aligned}
&\frac{d}{dt}\|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})\|_2^2 + \frac{\alpha}{2}\|(\nabla\nabla_h \mathbf{u}, \nabla\nabla_h \mathbf{w}, \nabla\nabla_h \mathbf{b})\|_2^2 + 2\kappa\|\nabla_h(\nabla\cdot\mathbf{w})\|_2^2 + 2\chi\|\nabla_h \mathbf{w}\|_2^2 \\
&\leq C\|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})\|_2^2\|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})\|_2^2.
\end{aligned}
$$

Assim, pelo Lema de Gronwall, inferimos

$$
\|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})(\cdot,t)\|_2^2 \leq C\|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})(\cdot,\delta)\|_2^2 e^{\int_\delta^T \|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})(\cdot,s)\|_2^2\, ds}, \quad \forall\, t \in [\delta, T^*).
$$

Por (1.1), temos que

$$
\|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})(\cdot,t)\|_2^2 \leq C, \quad \forall\, t \in [\delta, T^*). \qquad (3.35)
$$

Já vimos, também na demonstração do Teorema 3.1, que

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})\|_2^2 + \alpha\|(\Delta \mathbf{u}, \Delta \mathbf{w}, \Delta \mathbf{b})\|_2^2 + \kappa\|\nabla(\nabla\cdot\mathbf{w})\|_2^2 + \chi\|\nabla \mathbf{w}\|_2^2 \\
&\leq C\int_{\mathbb{R}^3} |(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})||(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})|^2\, dx.
\end{aligned}
$$

Assim, por (3.35) e pelas Desigualdades de Hölder, Gagliardo-Nirenberg e Young, chegamos a

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^2+\alpha\|(\Delta\mathbf{u},\Delta\mathbf{w},\Delta\mathbf{b})\|_2^2+\kappa\|\nabla\nabla\cdot\mathbf{w}\|_2^2+\chi\|\nabla\mathbf{w}\|_2^2\\
&\leq C\|(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})\|_2\ \|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_4^2\\
&\leq C\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^{\frac{1}{2}}\ \|(\Delta\mathbf{u},\Delta\mathbf{w},\Delta\mathbf{b})\|_2^{\frac{3}{2}}\\
&\leq C\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^2+\frac{\alpha}{2}\|(\Delta\mathbf{u},\Delta\mathbf{w},\Delta\mathbf{b})\|_2^2.
\end{aligned}$$

Consequentemente, obtemos

$$\frac{d}{dt}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^2+\alpha\|(\Delta\mathbf{u},\Delta\mathbf{w},\Delta\mathbf{b})\|_2^2+2\kappa\|\nabla(\nabla\cdot\mathbf{w})\|_2^2+2\chi\|\nabla\mathbf{w}\|_2^2\leq C\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^2.$$

Logo, pelo Lema de Gronwall, temos que

$$\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\cdot,t)\|_2^2\leq\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\cdot,\delta)\|_2^2e^{C\int_\delta^T ds},\quad\forall\,\delta\leq t<T^*.$$

Isto é uma contradição (ver desigualdade de Leray em [3]). Isto finaliza a prova do Teorema 3.2. $\square$

3.3 Critério de Regularidade Envolvendo $(\nabla u_3,\nabla\mathbf{w},\nabla\mathbf{b})(\cdot,t)$

Nesta seção, estenderemos um critério de regularidade estabelecido para uma solução fraca $\mathbf{u}(\cdot,t)$ das equações de Navier-Stokes (3), o qual garante a suavidade desta solução em $[0,T]$, quando adota-se a seguinte hipótese:

$$\nabla u_3(\cdot,t)\in L^p(0,T;L^q(\mathbb{R}^3)),\quad\frac{2}{p}+\frac{3}{q}=\frac{3}{2}+\frac{3}{4q},\frac{3}{2}<q<3.$$

(Para mais detalhes ver [68]). Nesta dissertação, conseguimos um resultado semelhante para uma solução fraca $(\mathbf{u},\mathbf{w},\mathbf{b})(\cdot,t)$ das equações magneto-micropolares (1), quando assumimos que

$$(\nabla u_3,\nabla\mathbf{w},\nabla\mathbf{b})(\cdot,t)\in L^p(0,T;L^q(\mathbb{R}^3));\quad\frac{2}{p}+\frac{3}{q}=\frac{3}{2}+\frac{3}{q},\ \frac{3}{2}<q<3.$$

Primeiramente, provaremos uma desigualdade que será útil na prova do resultado principal desta seção.

Lema 3.3 (ver [68]). *Sejam $\frac{3}{2}<q<3$ e $f,g\in C_c^1(\mathbb{R}^3)$. Então a seguinte desigualdade é verdadeira:*

$$\int_{\mathbb{R}^3}[f(x)]^2[g(x)]^2\,dx\leq C\|\nabla f\|_q^2\|g\|_2^{\frac{2(2q-3)}{q}}\ \|\nabla_h g\|_2^{\frac{2(3-q)}{q}},$$

onde C é uma constante positiva que depende somente de q.

Demonstração. Note que, pela Desigualdade de Hölder, temos

$$\begin{aligned}\int_{\mathbb{R}^3} f^2 g^2\, dx &\le \int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} |f|^{\frac{2q}{3-q}}\, dx_1 dx_2 \right)^{\frac{3-q}{q}} \left(\int_{\mathbb{R}^2} |g|^{\frac{2q}{2q-3}}\, dx_1 dx_2 \right)^{\frac{2q-3}{q}} dx_3 \\ &\le \max_{x_3 \in \mathbb{R}} \left(\int_{\mathbb{R}^2} |f(x_1,x_2,x_3)|^{\frac{2q}{3-q}}\, dx_1 dx_2 \right)^{\frac{3-q}{q}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} |g|^{\frac{2q}{2q-3}}\, dx_1 dx_2 \right)^{\frac{2q-3}{q}} dx_3.\end{aligned}$$

Vamos agora estimar o máximo encontrado do lado direito da desigualdade acima. Assim sendo, pelo Teorema Fundamental do Cálculo, temos

$$f^{\frac{2q}{3-q}}(x) = \int_{-\infty}^{x_3} \partial_3 f^{\frac{2q}{3-q}}(x_1,x_2,s)\, ds \le C \int_{\mathbb{R}} |f(x_1,x_2,x_3)|^{\frac{3(q-1)}{3-q}} |\partial_3 f(x_1,x_2,x_3)|\, dx_3.$$

Consequentemente,

$$\max_{x_3 \in \mathbb{R}} \left(\int_{\mathbb{R}^2} |f(x_1,x_2,x_3)|^{\frac{2q}{3-q}}\, dx_1 dx_2 \right)^{\frac{3-q}{q}} \le C \left(\int_{\mathbb{R}^3} |f|^{\frac{3(q-1)}{3-q}} |\partial_3 f|\, dx \right)^{\frac{3-q}{q}}.$$

Aplicando a Desigualdade de Gagliardo-Nirenberg

$$\|v\|_{\frac{2q}{2q-3}} \le C \|v\|_2^{\frac{2q-3}{q}} \|\nabla_h v\|_2^{\frac{3-q}{q}},$$

e a Desigualdade de Hölder, chegamos a

$$\begin{aligned}\int_{\mathbb{R}^3} f^2 g^2\, dx &\le C \left(\int_{\mathbb{R}^3} |f|^{\frac{3(q-1)}{3-q}} |\partial_3 f|\, dx \right)^{\frac{3-q}{q}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} |g|^2\, dx_1 dx_2 \right)^{\frac{2q-3}{q}} \left(\int_{\mathbb{R}^2} |\nabla_h g|^2\, dx_1 dx_2 \right)^{\frac{3-q}{q}} dx_3 \\ &\le C \|f\|_{\frac{3q}{3-q}}^{\frac{3(q-1)}{q}} \|\partial_3 f\|_q^{\frac{3-q}{q}} \|g\|_2^{\frac{2(2q-3)}{q}} \|\nabla_h g\|_2^{\frac{2(3-q)}{q}}.\end{aligned}$$

Por fim, por aplicar o Lema 2.1, encontramos

$$\|f\|_{\frac{3q}{3-q}}^{\frac{3(q-1)}{q}} \le C \|\partial_1 f\|_q^{\frac{q-1}{q}} \|\partial_2 f\|_q^{\frac{q-1}{q}} \|\partial_3 f\|_q^{\frac{q-1}{q}}.$$

Isto completa a prova do lema em questão. □

Agora estamos prontos para enunciar o resultado principal desta seção.

Teorema 3.3. *Seja* $(\mathbf{u_0}, \mathbf{w_0}, \mathbf{b_0}) \in L^2(\mathbb{R}^3)$ *com* $\nabla \cdot \mathbf{u_0} = \nabla \cdot \mathbf{b_0} = 0$ *e* $T > 0$. *Assuma que* $(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t)$ *é uma solução fraca das equações magneto-micropolares* (1) *em* $[0,T]$ *com condição* *inicial* $(\mathbf{u_0}, \mathbf{w_0}, \mathbf{b_0})$. *Se*

$$(\nabla u_3, \nabla \mathbf{w}, \nabla \mathbf{b})(\cdot, t) \in L^p(0, T; L^q(\mathbb{R}^3)); \quad \frac{2}{p} + \frac{3}{q} = \frac{3}{2} + \frac{3}{q}, \ \frac{3}{2} < q < 3, \tag{3.36}$$

então a solução $(\mathbf{u}, \mathbf{w}, \mathbf{b})$ *é suave em* $\mathbb{R}^3 \times (0, T)$.

Demonstração. A mesma discussão estabelecida no início da demonstração do Teorema 3.1 será considerada aqui. Assim sendo, como $(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})(\cdot, t) \in L^2(0, T; L^2(\mathbb{R}^3))$ (ver definição de solução fraca), então existe $\Gamma \in [t_0, T^*)$ tal que

$$\|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})(\cdot, \Gamma)\|_2 \le C \quad e \quad \int_\Gamma^{T^*} \|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})(\cdot, t)\|_2 \, dt < \delta, \tag{3.37}$$

onde $\delta > 0$ será estabelecido a seguir.

Relembrando a deigualdade (3.34), concluímos que

$$\begin{aligned} &\frac{d}{dt}\|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})\|_2^2 + 2\alpha\|\nabla\nabla_h \mathbf{u}, \nabla\nabla_h \mathbf{w}, \nabla\nabla_h \mathbf{b})\|_2^2 + 2\kappa\|\nabla_h(\nabla \cdot \mathbf{w})\|_2^2 + 2\chi\|\nabla_h \mathbf{w}\|_2^2 \\ &\le C \int_{\mathbb{R}^3} |(u_3, \mathbf{w}, \mathbf{b})||(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})||(\nabla\nabla_h \mathbf{u}, \nabla\nabla_h \mathbf{w}, \nabla\nabla_h \mathbf{b})| \, dx, \end{aligned}$$

onde $\alpha = \min\{\mu, \nu, \gamma\}$. Consequentemente, pela Desigualdade de Young, temos

$$\begin{aligned} &\frac{d}{dt}\|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})\|_2^2 + 2\alpha\|(\nabla\nabla_h \mathbf{u}, \nabla\nabla_h \mathbf{w}, \nabla\nabla_h \mathbf{b})\|_2^2 + 2\kappa\|\nabla_h(\nabla \cdot \mathbf{w})\|_2^2 + 2\chi\|\nabla_h \mathbf{w}\|_2^2 \\ &\le C \int_{\mathbb{R}^3} |(u_3, \mathbf{w}, \mathbf{b})|^2|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})|^2 \, dx + \alpha\|(\nabla\nabla_h \mathbf{u}, \nabla\nabla_h \mathbf{w}, \nabla\nabla_h \mathbf{b})\|_2^2. \end{aligned}$$

Portanto, pelo Lema 3.3, chegamos a

$$\begin{aligned} &\frac{d}{dt}\|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})\|_2^2 + \alpha\|(\nabla\nabla_h \mathbf{u}, \nabla\nabla_h \mathbf{w}, \nabla\nabla_h \mathbf{b})\|_2^2 + 2\kappa\|\nabla_h(\nabla \cdot \mathbf{w})\|_2^2 + 2\chi\|\nabla_h \mathbf{w}\|_2^2 \\ &\le C\|(\nabla u_3, \nabla \mathbf{w}, \nabla \mathbf{b})\|_q^2\|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})\|_2^{\frac{2(2q-3)}{q}}\|(\nabla\nabla_h \mathbf{u}, \nabla\nabla_h \mathbf{w}, \nabla\nabla_h \mathbf{b})\|_2^{\frac{2(3-q)}{q}}. \end{aligned}$$

Com isso, pela Desigualdade de Young, encontramos

$$\begin{aligned} &\frac{d}{dt}\|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})\|_2^2 + \alpha\|(\nabla\nabla_h \mathbf{u}, \nabla\nabla_h \mathbf{w}, \nabla\nabla_h \mathbf{b})\|_2^2 + 2\kappa\|\nabla_h(\nabla \cdot \mathbf{w})\|_2^2 + 2\chi\|\nabla_h \mathbf{w}\|_2^2 \\ &\le C\|(\nabla u_3, \nabla \mathbf{w}, \nabla \mathbf{b})\|_q^{\frac{2q}{2q-3}}\|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})\|_2^2 + \frac{\alpha}{2}\|(\nabla\nabla_h \mathbf{u}, \nabla\nabla_h \mathbf{w}, \nabla\nabla_h \mathbf{b})\|_2^2. \end{aligned}$$

Assim sendo, podemos escrever

$$\frac{d}{dt}\|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})\|_2^2 + \frac{\alpha}{2}\|(\nabla\nabla_h \mathbf{u}, \nabla\nabla_h \mathbf{w}, \nabla\nabla_h \mathbf{b})\|_2^2 + 2\kappa\|\nabla_h(\nabla\cdot\mathbf{w})\|_2^2 + 2\chi\|\nabla_h \mathbf{w}\|_2^2$$
$$\leq C\|(\nabla u_3, \nabla\mathbf{w}, \nabla\mathbf{b})\|_q^{\frac{2q}{2q-3}}\|(\nabla\mathbf{u}, \nabla\mathbf{w}, \nabla\mathbf{b})\|_2^2.$$

Integrando sobre $[\Gamma, t]$, onde $t < T^*$, a desigualdade acima, chegamos a

$$\|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})(\cdot,t)\|_2^2 + \frac{\alpha}{2}\int_\Gamma^t \|(\nabla\nabla_h \mathbf{u}, \nabla\nabla_h \mathbf{w}, \nabla\nabla_h \mathbf{b})(\cdot,s)\|_2^2\,ds + 2\kappa\int_\Gamma^t \|\nabla_h(\nabla\cdot\mathbf{w})(\cdot,s)\|_2^2\,ds$$
$$+ 2\chi\int_\Gamma^t \|\nabla_h \mathbf{w}(\cdot,s)\|_2^2\,ds \leq C + C\int_\Gamma^t \|(\nabla u_3, \nabla\mathbf{w}, \nabla\mathbf{b})(\cdot,s)\|_q^{\frac{2q}{2q-3}}\|(\nabla\mathbf{u}, \nabla\mathbf{w}, \nabla\mathbf{b})(\cdot,s)\|_2^2\,ds.$$

Definamos,

$$\mathcal{I}_\Gamma^2(t) = \sup_{\Gamma\leq\tau\leq t}\|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})(\cdot,\tau)\|_2^2 + \int_\Gamma^t \|(\nabla\nabla_h \mathbf{u}, \nabla\nabla_h \mathbf{w}, \nabla\nabla_h \mathbf{b})(\cdot,\tau)\|_2^2\,d\tau$$

e também

$$\mathcal{J}_\Gamma^2(t) = \sup_{\Gamma\leq\tau\leq t}\|(\nabla\mathbf{u}, \nabla\mathbf{w}, \nabla\mathbf{b})(\cdot,\tau)\|_2^2 + \int_\Gamma^t \|(\Delta\mathbf{u}, \Delta\mathbf{w}, \Delta\mathbf{b})(\cdot,\tau)\|_2^2\,d\tau.$$

Assim, pela hipótese (3.36) e por (1.1), temos

$$\mathcal{I}_\Gamma^2(t) \leq C + C\mathcal{J}_\Gamma^{\frac{3}{2}}(t)\left(\int_0^T \|(\nabla u_3, \nabla\mathbf{w}, \nabla\mathbf{b})\|_q^{\frac{8q}{3(2q-3)}}\,d\tau\right)^{\frac{3}{4}}\left(\int_0^T \|(\nabla\mathbf{u}, \nabla\mathbf{w}, \nabla\mathbf{b})\|_2^2\,d\tau\right)^{\frac{1}{4}}$$
$$\leq C + C\mathcal{J}_\Gamma^{\frac{3}{2}}(t).$$

Por conseguinte,

$$\mathcal{I}_\Gamma^2(t) \leq C + C\mathcal{J}_\Gamma^{\frac{3}{2}}(t), \quad \forall\,\Gamma \leq t < T^*. \tag{3.38}$$

Já vimos, na demonstração do Teorema 3.1, que

$$\frac{1}{2}\frac{d}{dt}\|(\nabla\mathbf{u}, \nabla\mathbf{w}, \nabla\mathbf{b})\|_2^2 + \alpha\|(\Delta\mathbf{u}, \Delta\mathbf{w}, \Delta\mathbf{b})\|_2^2 + \kappa\|\nabla(\nabla\cdot\mathbf{w})\|_2^2 + \chi\|\nabla\mathbf{w}\|_2^2$$
$$\leq C\int_{\mathbb{R}^3}|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})||(\nabla\mathbf{u}, \nabla\mathbf{w}, \nabla\mathbf{b})|^2 dx$$
$$\leq C\|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})\|_2\|(\nabla\mathbf{u}, \nabla\mathbf{w}, \nabla\mathbf{b})\|_4^2,$$

onde na última passagem utilizamos a Desigualdade de Hölder. Assim, pela Desigualdade de

Interpolação, pelo Lema 2.1 e a Desigualdade de Young, obtemos

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^2+\alpha\|(\Delta\mathbf{u},\Delta\mathbf{w},\Delta\mathbf{b})\|_2^2+\kappa\|\nabla(\nabla\cdot\mathbf{w})\|_2^2+\chi\|\nabla\mathbf{w}\|_2^2\\
&\leq C\|(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})\|_2\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^{\frac{1}{2}}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_6^{\frac{3}{2}}\\
&\leq C\|(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})\|_2\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^{\frac{1}{2}}\|(\nabla\nabla_h\mathbf{u},\nabla\nabla_h\mathbf{w},\nabla\nabla_h\mathbf{b})\|_2\|(\Delta\mathbf{u},\Delta\mathbf{w},\Delta\mathbf{b})\|_2^{\frac{1}{2}}\\
&\leq C\|(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})\|_2^{\frac{4}{3}}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^{\frac{2}{3}}\|(\nabla\nabla_h\mathbf{u},\nabla\nabla_h\mathbf{w},\nabla\nabla_h\mathbf{b})\|_2^{\frac{4}{3}}\\
&\quad+\frac{\alpha}{2}\|(\Delta\mathbf{u},\Delta\mathbf{w},\Delta\mathbf{b})\|_2^2.
\end{aligned}
$$

Assim, podemos inferir que

$$
\begin{aligned}
&\frac{d}{dt}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^2+\alpha\|(\Delta\mathbf{u},\Delta\mathbf{w},\Delta\mathbf{b})\|_2^2+2\kappa\|\nabla(\nabla\cdot\mathbf{w})\|_2^2+2\chi\|\nabla\mathbf{w}\|_2^2\\
&\leq C\|(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})\|_2^{\frac{4}{3}}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^{\frac{2}{3}}\|(\nabla\nabla_h\mathbf{u},\nabla\nabla_h\mathbf{w},\nabla\nabla_h\mathbf{b})\|_2^{\frac{4}{3}}.
\end{aligned}
$$

Integrando sobre $[\Gamma,t]$, onde $t<T^*$, a desigualdade acima, obtemos

$$
\begin{aligned}
&\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\cdot,t)\|_2^2+\alpha\int_\Gamma^t\|(\Delta\mathbf{u},\Delta\mathbf{w},\Delta\mathbf{b})(\cdot,s)\|_2^2\,ds+2\kappa\int_\Gamma^t\|\nabla(\nabla\cdot\mathbf{w})(\cdot,s)\|_2^2\,ds\\
&+2\chi\int_\Gamma^t\|\nabla\mathbf{w}(\cdot,s)\|_2^2\,ds\leq C+C\int_\Gamma^t\|(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})(\cdot,s)\|_2^{\frac{4}{3}}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\cdot,s)\|_2^{\frac{2}{3}}\\
&\times\|(\nabla\nabla_h\mathbf{u},\nabla\nabla_h\mathbf{w},\nabla\nabla_h\mathbf{b})(\cdot,s)\|_2^{\frac{4}{3}}\,ds.
\end{aligned}
$$

Desta forma, pela Desigualdade de Hölder e (3.38), chegamos a

$$
\begin{aligned}
\mathcal{J}_\Gamma^2(t)&\leq C+C\mathcal{I}_\Gamma^{\frac{4}{3}}(t)\left(\int_\Gamma^t\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^2\,ds\right)^{\frac{1}{3}}\left(\int_\Gamma^t\|(\nabla\nabla_h\mathbf{u},\nabla\nabla_h\mathbf{w},\nabla\nabla_h\mathbf{b})\|_2^2\,ds\right)^{\frac{2}{3}}\\
&\leq C+C\mathcal{I}_\Gamma^{\frac{8}{3}}(t)\left(\int_\Gamma^t\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^2\,ds\right)^{\frac{1}{3}}\\
&\leq C+C[C+C\mathcal{J}_\Gamma^{\frac{3}{4}}(t)]^{\frac{8}{3}}\left(\int_\Gamma^t\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^2\,ds\right)^{\frac{1}{3}}\\
&\leq C+[C+C\mathcal{J}_\Gamma^2(t)]\left(\int_\Gamma^{T^*}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^2\,ds\right)^{\frac{1}{3}}.
\end{aligned}
$$

Por (3.37), temos que

$$
\int_\Gamma^{T^*}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\cdot,s)\|_2^2\,ds<\delta,
$$

onde $\delta = \frac{1}{8C^3}$. Logo,

$$\mathcal{J}_\Gamma^2(t) \leq C, \quad \forall\, t \in [\Gamma, T^*).$$

E isto conclui a prova desse teorema. $\square$

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